Developing Common Sense in Teaching Mathematics

Jeremy Kilpatrick

Abstract

In “Mathematics Education and Common Sense,” Keitel and Kilpatrick (2005) observe that mathematics teaching too often privileges a study of the discipline over an explicit development of learners’ common sense. At the same time, the hidden curriculum of mathematics teaching ends up creating a common sense for learners that teachers seldom share. What is true for teachers and learners can also hold for mathematicians and teachers. In this paper, I explore some examples from recent disputes—the so-called math wars—in which the presumed commonsense view of school mathematics being proposed by mathematicians is not seen as common sense by teachers. One means of reaching common ground in the math wars, therefore, might be for mathematicians and teachers alike to educate their common sense.

The mathematician George Pólya was fond of pointing out that although we can usually rely on our intuition for ideas on how to solve a mathematics problem, sometimes our intuition lets us down. We need to, in his words, “educate our intuition.” In mathematics education, intuition is often seen as akin to common sense (Fischbein, 1987; Freudenthal, 1991), and our common sense needs to be educated, too. Common sense is both an individual possession and a social construction. It helps us learn, do, and teach mathematics, and it also can hinder all those processes.

School mathematics has historically attempted to mirror what has been seen as the abstract, context-free, universal nature of academic mathematics. Consequently, mathematics teaching has tended to concentrate on the promotion of skill in handling routine numerical, algebraic, and geometric operations divorced from meaningful contexts or realistic applications. Far from drawing on, let alone developing, learners’ commonsense notions of quantity and space, instruction seeks out the rarefied realm of abstraction, formalism, and generality. With few exceptions, learners respond to such instruction with boredom and indifference.

The common sense developed by learners thereby becomes a perspective that is less about mathematics than about how to cope in the mathematics classroom. A curriculum that is all but hidden from the teacher educates learners as to the minimum amount of effort the teacher will accept, what will count as evidence of interest and learning, answers the teacher expects to hear, and what ought not to be said or done. Each group then works within its own common sense, with teachers pitting mathematics against what they see as ignorance and common misconceptions, and learners pitting their wits against what they see as senseless activity. Teachers and learners are separated not merely by faulty communication but also by different views of common sense and school mathematics.

Two other groups that can be separated in a similar fashion are mathematicians and schoolteachers. The teaching of school mathematics involves the interactions among a teacher, some learners, and some mathematical subject matter in which they are jointly engaged. Mathematicians typically see themselves, but not teachers, as experts on subject matter. Teachers typically see themselves, but not mathematicians, as experts on pedagogy and learners. In each case, there are opportunities to educate what the group sees as common sense about its expertise and the other group's lack of expertise.

Standards for School Mathematics

Some particularly rich examples of divergent perspectives on common sense and how they might be addressed can be found in controversies that developed in response to efforts by groups of mathematics teachers in North America to change school mathematics. Unlike the new mathematics movement of the 1950s to 1970s, which began with university mathematicians, the recent movement was initiated by a professional organization, the National Council of Teachers of Mathematics (NCTM). The NCTM is an organization of teachers and others concerned with mathematics education, and most of its members come from the United States and Canada, although growing numbers are from other countries. The NCTM currently has almost 100,000 members.

In 1980, the NCTM published *An Agenda for Action,* which made a number of recommendations, most prominently that problem solving be the focus of school mathematics, with basic skills defined as more than computation. The *Agenda* was NCTM's way of providing direction to the field, but it was also its first major effort to influence public policy. The *Agenda* was both well received and ultimately rather influential in national education policy. The organization began to realize that it had an important political role to play. Then in 1984, the NCTM Board of Directors appointed a task force to plan the development of comprehensive guidelines for the kindergarten to Grade 12 school mathematics program. The first project was the writing of *Curriculum and Evaluation Standards for School Mathematics,* published in 1989. Standards for teaching mathematics were dealt with in a later project, leading to *Professional Standards for Teaching Mathematics* in 1991. Four years after that, *Assessment Standards for School Mathematics* was published. These documents, particularly the first, helped launch what became known as the standards movement. The documents attempted to provide a vision of mathematical literacy for today's world.

Through the first half of the 1990s, the reaction to the standards was almost entirely positive. Publishers began to call their textbooks "standards-based." Curriculum development projects, first at the middle school level and then at the high school level, were funded by the U.S. National Science Foundation to develop new instructional materials. States and local districts began to "align" their mathematics curricula with the NCTM standards. The few discussions of the mathematics standards in the media did not take issue with what they said.

The Math Wars

Gradually, however, a backlash began to form. The NCTM was charged with promoting a movement labeled *whole math,* a term chosen so that it could be lumped together with *whole* *language* methods of teaching reading and both then characterized as efforts to subvert education. Another label for the standards-based reform was the *new-new math*, which indicated that it was somehow a successor to the new math of the 1960s, an approach also seen as discredited. Groups of parents and mathematicians were formed to work against standard-based changes in school mathematics. Of these groups, the oldest and most visible is Mathematically Correct, which runs an up-to-date and informative Web site <http://ourworld.compuserve.com/homepages/mathman/index.htm> that presents the anti-reform position in detail. (For details of the movement in California, the backlash there, and its consequences, see Wilson, 2003).

Critics began to use terms like *fuzzy math* to characterize almost anything done in the name of standards-based change. Defenders of that change then accused critics of wanting to return to a *parrot math* curriculum. The media related horror stories of children wasting their time in misguided "explorations" and not learning basic facts. In January 1998, Richard Riley, the U.S. Secretary of Education, weighed into the controversy, calling for a cease fire in what he termed the "math wars"—reviving a term that had been used almost 40 years before to characterize the new math reform efforts (DeMott, 1962, ch. 9).

In response, and to take advantage of the first 10 years of experience with and reactions to the standards, NCTM published in 2000 its *Principles and Standards for School Mathematics.* That document brought together and updated standards for the mathematics classroom, combining curriculum, teaching, and assessment. Because mathematicians had been at the forefront of many of the criticisms of its earlier standards documents, the NCTM included more mathematicians on the writing groups for the 2000 document and made stronger efforts to get prepublication reviews by mathematicians. Much criticism, however, continued to be heard, indicating that there were still some topics on which perspectives diverged.

Divergent Common Sense

Definitions

One topic on which the common sense of mathematicians appears to differ from that of teachers is the role of definitions in teaching mathematics. Mathematicians tend to regard the presentation of a precise definition as essential when students are learning a new concept (Milgram, 2005, pp. 85–92). The definition should be "mathematically accurate" (Mathematics Standards Study Group [MSSG], 2005). For example, in introducing the topic of function, the set-theoretic Dirichlet-Bourbaki definition might be provided so that learners would have an unambiguous criterion to apply. In contrast, schoolteachers tend to be wary of providing a formal definition before learners have some idea of what is being defined (and for which they have developed a personal definition). In the common sense of teachers, the learner needs an image of the concept (Tall & Vinner, 1981; Vinner, 1983) before a formal definition can be understood. Furthermore, teachers are often content to work with a provisional definition until learners have explored the concept and have become familiar with various examples and nonexamples.

Algorithms

So that learners can perform computations, they need to learn algorithms for the operations. Algorithms differ greatly in transparency, efficiency, generality, and precision (Kilpatrick, Swafford, & Findell, 2001, p. 103), and learners need to understand how to find an appropriate balance among these characteristics. A teacher may encourage learners to work with a transparent but relatively inefficient algorithm, for example, so that they can see how the algorithm operates. The teacher may even allow learners to construct and use their own algorithms before moving on to one that is more precise and general. In contrast, mathematicians typically see little value in learning any algorithm that is not what they consider to be a “standard” paper-and-pencil algorithm. They argue against the adoption of “student developed algorithms” (Milgram, 2005, p. 178). It seems only common sense to learn one algorithm that works efficiently every time rather than using a half-baked procedure. For teachers, however, it may not be so clear which algorithm ought to be taken as standard, particularly when the learners come from families in which the adults went to school in different systems and were taught different algorithms. The teachers’ common sense leads them to make sure that learners know and can use “an algorithm that is general and reasonably efficient” (Kilpatrick et al., p. 414) but not necessarily the same algorithm that a mathematician might label *standard*.

Technology

Technology, and in particular the use of calculators, is a topic on which the common sense within the community of mathematicians and the community of teachers is almost as mixed as it is between the two groups. One line of common sense, promoted by many but far from all teachers, argues for the extensive use of technology in school mathematics because of the power it provides learners to visualize and explore complex situations dynamically. Another line of common sense, however, primarily from mathematicians but also from teachers, argues that learners become “too dependent” on technology when it is used too extensively in school. Learners reach for technology when they should be using their memory and their reasoning abilities.

Calculators appear to pose a particularly vexing issue. A recent statement from a group of mathematicians who were addressing the issue of giving elementary school students practice in performing multi-digit arithmetic operations, said, “The only role for calculators in this process is to check answers computed by hand” (MSSG, 2005, p. 4). The 1989 NCTM standards document had seemed to be promoting calculator use when it said, “Appropriate calculators should be available to all students at all times” (p. 8). But the May 2005 NCTM position statement entitled “Computation, Calculators, and Common Sense” appeared to take a more middle-of-the-road stance, saying, “The teacher should help students learn when to use a calculator and when not to, when to use pencil and paper, and when to do something in their heads.” Calculator use appears to be a topic on which common sense has yet to congeal across and within communities.

Statistics

Many teachers in North America have embraced the inclusion of data analysis and statistics in the school curriculum because it allows them to engage students in interesting class activities, including the construction of mathematical models for the data. They find that students enjoy using mathematics to investigate realistic situations involving statistics (Kilpatrick, Hancock, Mewborn, & Stallings, 1996). The 2000 NCTM standards document proposes data analysis and probability as a standard throughout the grades from prekindergarten to Grade 12. Some mathematicians, however, take a more jaundiced view of the introduction of statistics into the school curriculum. They worry about schoolchildren wasting their time making histograms of data they have gathered when they could be learning arithmetic. They see statistics in school mathematics as involving little serious work. At a March 2006 conference on issues in school mathematics, a prominent mathematician told the teachers present that they should leave the teaching of statistics to the colleges and universities "where we know how to do it right." Their common sense tells teachers that learners respond well to statistics as part of their mathematics education. Their common sense tells mathematicians that statistics is difficult to teach well and may take up space in the school curriculum better left to important mathematics.

Mathematics for All

One of the most curious themes on which mathematicians appear to diverge in their commonsense views from the teachers in NCTM concerns the latter's efforts to promote mathematics for all. The NCTM 2000 standards document says, "Equity … demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students" (p. 12). The slogan on the NCTM Web site (http://www.nctm.org) proclaims "More and Better Mathematics for All Students." Nonetheless, at least some mathematicians question that idea. Martin Gardner (1998), reviewing a commercial algebra textbook, an NCTM yearbook on equity, and a public television videotape series on mathematics, argues against the NCTM's effort to change school mathematics, calling it both "fuzzy math" and "the new new math." He complains that it "is heavily influenced by multiculturalism, environmentalism, and feminism" (p. 9). Although he rightly complains about the absence of recreational mathematics in the materials under review as well as their over-reliance on cultural artifacts of questionable value or validity, he seems almost to resent the notion that school mathematics might be made more accessible to more students. Such critics of the NCTM's efforts treat school mathematics as akin to a finite resource to be distributed only to the deserving, those willing to work hard to attain it.

Seeking Common Ground

In December 2004, Richard Schaar, a mathematician and senior vice president of Texas Instruments, invited two mathematicians, James Milgram of Stanford and Wilfried Schmid of Harvard, and three mathematics educators, Deborah Loewenberg Ball of the University of Michigan, Joan Ferrini-Mundy of Michigan State University, and me, to a so-called peace summit in Washington, DC, where we attempted to find common ground in the math wars. After a second meeting at the offices of the Mathematical Association of America (MAA) in June 2005, with numerous e-mail exchanges in between, the group posted a statement on the MAA Web site, and it was published in the *AMS Notices* (Ball et al., 2005). The purpose of the statement was to demonstrate that there was agreement across groups on some of the points being debated in the math wars.

A response by Tony Ralston (2006) called the effort “a valuable exercise” but also concluded that the statement was unexceptional, bland, and ambiguous. It did not, he said, address major points of contention, including curriculum and technology. He argued that before any nontrivial consensus on issues of school mathematics could be achieved, there first needed to be “a level of respect in both communities for the other that will mean that inevitable disagreements need not erupt into shouting matches.”

To continue the search for common ground, a meeting of approximately 75 mathematicians, teachers, and other mathematics educators was held at Indiana University-Purdue University Indianapolis on March 2–5, 2006. The purpose was to explore issues in school mathematics on which common ground might be achieved, spreading the conversation beyond the small group that had begun it. Five groups were formed to address the topics of algebra, algorithms, probability and statistics, technology, and teacher preparation. Reports from the groups were posted on the MAA Common Ground Web page (http://www.maa.org/common-ground). Those reports, although far from polished or definitive, indicated that there was, indeed, considerable agreement in the various communities concerned with the teaching of school mathematics.

Developing Common Sense

The Common Ground report and its aftermath, like that of the National Research Council’s Mathematics Learning Study (Kilpatrick et al., 2001) or the report of the RAND Mathematics Study Panel (2003), demonstrate that what appear to be divergent commonsense views of various groups concerned with school mathematics can be reconciled, given the right circumstances. Members of different groups can come together and educate their collective common sense.

What does it take? For one thing, it takes time. The 4-page Common Ground statement took 7 months. The report of the Mathematics Learning Study took 2 years. The RAND report took almost 3 years. People need an opportunity to get acquainted with each others’ views and understand how they are thinking about the issues. As Gardner’s (2006) critique suggested, they need to develop a climate of mutual respect and honest exchange of opinion before they can move forward.

An important feature of these efforts appears to be the engagement by all participants in the production of a written report. That task requires that people listen to one another carefully, ask for clarification or examples when a point is not clear, and formulate language that all can agree on. Working collectively in this way gets people away from the more extreme language that they might use when giving a talk or posting a message on the Internet. It allows them to discover that the issues on which they disagree strongly are likely to be fewer than they had thought and that with sufficient discussion, they can reach some consensus.

Developing a collective common sense across the several communities concerned with the teaching of school mathematics will never be an easy matter. It will always require leadership, support, time, cooperation, communication, and good will. Nonetheless, it appears to be not only a worthwhile goal but an attainable one.

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