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A Note from the Editor

Dear readers and supporters of *TME*,

It's been a full year since the last issue of *TME*, in part because the former editor, Keith Leatham, was a hard act to follow. I want to thank Keith for his work on Volumes 10 and 11 and apologize for the lapse in issues. We are currently working to ensure that the transition between editors runs more smoothly than it has in the past! I also want to thank Andy Norton and Chris Drumm for their work in getting the current issue started. In addition, I want to acknowledge the reviewers for *TME* for this issue: Holly Anthony, Summer Brown, Serkan Hekimoglu, Dennis Hembree, Kelli Nipper, Kevin Nooney, Lisa Sheehy, David Stinson, and Shannon Umberger. Without their work this journal would not be possible. The editorial staff is always looking for more reviewers. If you are interested in reviewing for *TME*, please send an email to tme@coe.uga.edu. Please indicate if you have special interests in reviewing articles that address certain topics such as curriculum change, student learning, teacher education, or technology.

In keeping with the mission statement of *TME*, this issue represents diverse views about a range of topics from students and faculty in mathematics and mathematics education. The issue includes two forays by Andy Norton and Tom Kieren into an area not usually discussed in mathematics education: religion and mathematics. Jeff Knisley's proposed mathematics learning model is perhaps more familiar territory but may be different from some mathematics educators' views on student learning. The issue also includes Dorothy White's call for attention to equity issues in teacher education. Mark Boylan focuses on the "small" political nature of classroom communities in his examination of teacher questioning. In his article review, Kevin Nooney questions one author's take on the relationship between history of mathematics and mathematics education. The editorial staff invites responses from interested readers to any of the pieces in this issue.

I also want to alert all readers to the availability of *TME* online at www.ugamesa.org. As you may know, we are now encouraging readers to view *TME* electronically, although hard copy subscriptions for Volume 13 will continue to be $6 for individuals and $10 for institutions. If you currently receive a hard copy and instead would like to be notified by email when a new issue is available online, please send a message to tme@coe.uga.edu. Alternately, if you subscribe online and would like to receive a hard copy, please notify us via email or mail at the address below.

Finally, the editorial staff is calling for submissions, particularly from (but not limited to!) graduate students. *TME* conducts blind peer review and publishes a wide variety of manuscripts (see inside back cover.) We are interested in helping graduate students reach a broader audience with their work and in fostering communication among mathematics educators with a range of professional experience. Please submit!

Wishing you happy and productive reading, learning, teaching, and researching—
Amy Hackenberg

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About the cover
This mandala is the central design on a decorated copper tray from Kano (Nigeria) and is pictured on page 47 of Gerdes, P. (1999). *Geometry from Africa: Mathematical and educational explorations.* Washington, DC: The Mathematical Association of America.

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Guest Editorial…Preparing Preservice Teachers to Work in Diverse Mathematics Classrooms: A Challenge for All

Dorothy Y. White

Educating quality teachers to work effectively with diverse student populations is a challenge yet to be met by most educators. This challenge is imperative in mathematics education where people often accept disparities in achievement across various student backgrounds as being normal, natural, inevitable, explainable, or even acceptable (Secada, 1992). For example, a common misconception is that African American, Hispanic, White female, and poor students are not “mathematically inclined” and must work harder to succeed. In contrast, Asian and White male students are often seen to possess an innate ability to succeed in mathematics. Seeing students for what they know and can do mathematically, instead of what they look like or where they live, and understanding the similarities and differences in students’ world experiences are goals mathematics educators need to pursue when preparing the next generation of teachers.

As a mathematics educator, I strive to change my students’ image of mathematics learners and instill in them a belief that *all* children can learn mathematics. As an African American female mathematics educator, I also want my students to explore issues relating to gender, ethnicity, and class, and to investigate how these issues are enacted in mathematics classrooms and schools. In my experience, however, most preservice teachers have limited experience working and interacting with people different than themselves. Ninety-five percent of the preservice teachers I have taught are White middle-class females who attended schools with very little diversity in the student body, faculty, and administration. The few preservice teachers that have attended racially mixed schools have reported that they rarely interacted with people of different racial and ethnic backgrounds. As a result of their experiences, most of these preservice teachers envision themselves teaching in schools similar to the ones they attended where the students are like them.

Recent demographics suggest an increase in the likelihood that today’s preservice teachers will teach students whose ethnic backgrounds differ greatly from their own. The National Center for Educational Statistics (NCES, 2001) reported that minority students comprised 38 percent of the total US public school enrollment in 1999. These enrollments differ by region and range from 24 percent minority student enrollment in the Midwest to 47 and 45 percent in the West and South, respectively. The NCES further notes that the overall number of minority student enrollments is increasing and that Hispanic students are the fastest growing student population in US elementary and secondary schools. As the number of students of color entering our public schools increases, so does the need for teachers prepared to accommodate the mathematical needs of a wide variety of students.

Now more than ever, mathematics educators must meet the challenge of preparing preservice teachers to work effectively in diverse mathematics classrooms. As Sleeter (1997) recommends, the professional development of teachers at the preservice level should include research on the professional development of teachers, mathematics reform, and multicultural mathematics. The challenge to mathematics educators is to insure that as our students examine mathematics, teaching, and learning, they are provided with opportunities to: (1) engage in critical inquiry about equity issues, (2) gain experience in working with diverse students, and (3) explore the contributions of various cultures to the field of mathematics. These opportunities allow prospective teachers to examine school mathematics practices, reflect on their beliefs, and hopefully become agents of change.

Engage Preservice Teachers in Critical Inquiry about Equity Issues

Educating preservice teachers to work effectively with diverse students requires that they engage in critical inquiry about equity issues and about their influence on the mathematics education experiences of students; they need to know “the influence of students’ linguistic, ethnic, racial, and socioeconomic backgrounds and gender on learning mathematics” (NCTM, 1991, p. 144). Tracking, ability grouping, learning styles, testing, family-school connections, and technology are a few topics that should be addressed in mathematics teacher education courses. Many

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preservice teachers have benefited from common school practices like being placed in high-track mathematics classes. Helping them reflect on their experiences and think about the experiences of others challenges their conceptions and helps them question the status quo so they can become proactive in their classrooms and schools.

Many preservice teachers continue to view mathematics as a White male-dominated subject. As one of my preservice teachers stated on the first day of class last semester, "Everybody knows that boys are smarter than girls in math." Therefore, gender issues and the inequitable experiences of males and females in mathematics classrooms need to continuously be examined and discussed. Preservice teachers need to consider alternative explanations for the mathematical success and failure of students. Critically examining the experiences in school mathematics of students from various cultural backgrounds must be a central part of the discussion of equity issues in our mathematics education courses. These examinations must move beyond the experiences of Black and White students to include the experiences of students from different ethnic, linguistic, and religious backgrounds.

Being uncomfortable is an almost unavoidable part of the discussions but is necessary to grapple with important educational problems. Students need to feel part of the conversation rather than the object of a lecture. They need to feel free to articulate their views and to consider and respect the views of others.

Provide Experiences Working with Diverse Students

Preservice teachers need experience working with diverse students to examine how all students learn mathematics. Thus we must expand our definition of diversity and not limit it to students from different racial backgrounds. Diversity must include various religious, linguistic, educational, and socioeconomic backgrounds. The ideal experience would provide preservice teachers with an opportunity to work in schools that reflect diversity. They can explore the mathematical knowledge of various students, observe school practices, and observe the teaching of mathematics across the curriculum. However, a community project where students work with a local tutoring program, General Education Development (GED) program, low-income housing project, homeless shelter, church, synagogue, or mosque can also provide valuable experiences. The goal is for students to interact and work with people different than themselves.

In any setting, however, care must be given to make sure that students' misconceptions are not validated through these experiences. For example, if students work only with low-achieving African American students and high-achieving Asian students, these experiences can reaffirm their beliefs about which students can and cannot do mathematics. Therefore, a range of students should be chosen whenever possible. More importantly, class discussions that draw implications for the teaching and learning of mathematics must follow all out-of-class experiences.

Explore the Contributions of Various Cultures to the Field of Mathematics

In addition to critically examining equity issues and working with diverse students, preservice teachers need an opportunity to learn of the contributions from various cultures to the field of mathematics. There are several resources available that provide multicultural mathematics materials. For example, patterns and place value can be explored through African and Asian artifacts and games such as Mansala and Tangrams, respectively. Other games like Pachisi (India), Nyout (Korea) and Senet (Ancient Egypt) are excellent to develop children's' logical thinking. The old playground favorite of hop scotch, invented by the Romans, is yet another example to help young children explore counting strategies. These activities provide preservice teachers with a greater awareness of the contributions of various cultures to mathematics.

Educating preservice teachers to work effectively with diverse students requires that they learn to identify multicultural instructional materials and discuss their implications for teaching mathematics. Preservice teachers need help distinguishing between materials that Chappell and Thompson (2000) call "culturally contextual" from materials that are "culturally amendable." As they explain, culturally contextual materials have a cultural context that is essential to the message or material. In contrast, culturally amendable materials may have people of color as the main characters or be written in another language such as Spanish, but the message is not dependent on the culture. In other words, the characters portrayed can be of any ethnic persuasion. Learning the importance of each type of material and its implications for working with diverse student populations is important for preservice teachers' professional development.

Concluding Comments

Our teacher education candidates must leave our programs with an awareness of students' mathematical thinking and an ability to look toward and beyond students' gender, race, creed, and family income. Preparing preservice teachers to work in diverse mathematics classrooms rests with every mathematics educator. We must reflect on whether we are providing our students with the best education possible and increase our knowledge of multicultural mathematics education. Addressing diversity in our teacher education courses does not end with our students. We need to serve as resources, collaborating with our colleagues to expand an understanding of multicultural mathematics teaching, learning, and research. Educating preservice teachers to work in diverse mathematics classrooms may be a challenge, but it is also an exciting opportunity to learn about our students and to learn new approaches to reach *all* students mathematically. The next generation of teachers and students deserves to experience the wonders of mathematics teaching and learning while celebrating the diverse backgrounds of all.

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Teacher Questioning in Communities of Political Practice

Mark Boylan

Introduction

The central aim of this paper[1] is to contribute to a theoretical understanding of the nature of classrooms as communities of practice (Lave and Wenger, 1991; Wenger, 1998). Theories of participation in situated practices have been a rich way to understand the ways in which cognition and learning are social. Theories of participation give ontological and epistemological priority to action: they focus on what people do, but people are more than what they do. If we want to understand why we do what we do and how we experience our actions and those of others as meaningful, then other theoretical frameworks need to be drawn upon and theories of social practice need to be complemented by other perspectives.

An important theoretical tool, both in my interpretation of interview material and in the theoretical framework of this paper, is the concept of "life world" (Ashworth, 1997). Considering each student's life world means more than thinking about their individual experiences of the social practices of the classroom. Rather, it asserts the personal truth of each participant's situation. Thinking about students' perspectives on classroom interactions as giving insight into their life worlds is worthwhile for a number of reasons. Methodologically, it means listening to the students as credible informants. Theoretically, it allows us to make sense of the complexity of the classroom, the way in which there is one class but many worlds (Roth et al., 1999). Politically, accepting the validity of individuals' life worlds is congruent with a commitment to "deep democracy" (Mindell, 1995). However, I am not attempting a theoretical synthesis of theories of social practice and the concept of the life world; they have very different philosophical foundations that do not easily speak to each other. Rather, I am using both as maps that can help us to understand the same landscapes: one helps to map the way in which meaning is communal and shared, the other the way in which it is individual and diverse.

My focus here is on teacher questioning of students. Teacher questioning is a frequent, universal, and pervasive practice in school classrooms (Roth, 1996). The way teachers and students interact through questions both typifies and is productive of the fuller range of classroom practices and so offers insights into the nature of these social practices. In addition, the way in which students experience different forms of teacher questioning has not often been reported. The unreflective use of questions and some of the forms the practice can take has been criticized both on grounds of its effect on learning (Dillon, 1985; Dillon, 1988) and on children's affective experience of mathematics (Anderson, 2000; Anderson & Boylan, 2000).

One of my assertions in this paper is that mathematics classrooms are communities of political practice. My concern here is not with the "big" politics of mathematics education. This "big" politics is concerned with what aims and values are being promoted by policy and practice, what ideologies inform them, which social groups benefit, and how mathematics education helps to maintain and regenerate society.

There is another way in which education can be viewed as political and that is at the level of the interactions within each individual classroom community. The contestation of politics at the level of society as a whole develops due to conflicts over elements such as scarce resources, different interests, and different worldviews of members of society. Similar factors cause each classroom to be a political arena. Paying attention to this "small" politics is not simply about examining the ways in which policy, ideology and so on can be found in and affect individual classrooms. It is also about the politics that is produced within the individual community. For example, the big politics of gender relationships and identities are participated in and constructed within classrooms; the relationships between the genders are political. So also are the relationships *within* the gender groups between different individuals.

The labels "small" and "big" should not be taken to indicate the relative importance of these two arenas. Indeed, arguably it is at the level of the "small" that individual teachers, researchers and students have the most opportunity to act politically. There is a relationship between these two levels of politics. This relationship is a dialectical one in which contestation in each individual classroom is a reflection of, and framed by, what happens at the level of policy and ideology. However, in this paper I am primarily concerned with using a political lens to understand the individual classroom community.[2]

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Research material

To illustrate my claims I draw on material collected during an extended research project. The project was part of wider exploration of the nature of community within classrooms and an investigation into some of the limits and possibilities for transformation in secondary mathematics classrooms in the current UK context. I worked closely with a teacher of mathematics, Jill, in her first year of teaching. We focused on two of her classes. It is material from interviews and observation with one of these, Seven B, which I present here.[3] Seven B, a grade six class, were in the first year at secondary school at the time of the project. The school they attended, South School, is a large 11-18 school in London. Students' attainment is near national averages. Students come from a wide range of ethnic backgrounds and there are a significant proportion of refugee children and children whose home language is not English. The school has a gender imbalance with significantly more boys than girls. The composition of Seven B reflected the diversity and composition of the wider school population, with thirteen different ethnic backgrounds represented within the class. Approximately two-thirds of the students were boys and one-third girls. Year seven children are taught in mixed ability classes.

Research into different types of classrooms has tended to focus on paradigmatic cases. However, many classrooms share features of different typical cases (Boylan, Lawton & Povey, 2001). This characterization was true of Jill's classroom in which features of both more traditional and more inquiry-based practice could be found. However, questioning usually followed the common form of initiation, response and evaluation (Mehan, 1979). During the year, Jill attempted to enact more open and democratic practices. In particular she explored alternatives to children answering by putting their hands up and the teacher choosing who answers. The students' experiences of real or imagined questioning practices were one of the subjects of the interview material presented here.

Asking and answering questions and authority in the community

John, George, and Dave are ranking statements about possible situations that might occur during teacher questioning. The first dimension they have been asked to consider is how often the different events occur during their lessons and to put those that happen most often at the top and those that happen the least at the bottom.

John: [Reads] 'The teacher does not ask questions', that would be in Australia, way down

George: Maida Vale

[…]

Dave: Pluto

John: Canada

For these boys the idea of a teacher who did not ask questions of the class is almost unbelievable. It is so far removed from their experience that the physical metaphors they use are places on the other continents, the opposite side or the world, the far reaches of the solar system and somewhere just as far away, "Maida Vale".[4] The students' responses confirm the universal nature of teacher questions within their life worlds. All the other groups of students ranked "the teacher does not ask questions" in a similar place.

Although the students did and could ask questions, questioning by students has a different purpose:

Susan: 'we get to ask the question', no we never do

Jenny: exactly that's right at the bottom [tone of complaint]

Kerry: we get to ask the questions [wistfully or wondering tone]

Jenny: well sometimes like today we finished a, b, and c, we went down to the bottom and done like your own way that's like asking your own questions [referring to that day's mathematics lesson]. …The only time we get to ask questions is if we don't understand, the only time like [to] ask what kind of thing is this because it's like the teacher's job to do that to show us how to do everything.

"The teacher's job is to show us how to do everything" and to ask questions to ensure that the students know how to do it. The student's role is to answer questions and to "do it." When knowledge is viewed in this way then the holding of it confers authority on the holder. Given that it is almost taken for granted that the teacher has a monopoly on being the questioner, the means by which students can establish their authority is by answering questions. The students' purposes within Seven B, when asked about their responses to questioning practices, were often connected to status and authority and not to mathematical learning. Here, we see an example of the ways in which participation can hide a lack of engagement with mathematics (Denvir et al., 2001). Within Seven B, participation in teacher questioning practices often related to status and authority. Asking and answering questions was a means by which authority was claimed, established, and contested.

If we wish students to have greater mathematical authority, it means that control over interactions must be more equitably shared. An obvious way to share is to encourage students to ask meaningful questions. Interestingly, when asked what would be most helpful to their learning, the students invariably indicated asking questions. However, this action was also the possibility that they were most nervous about. It has been found that asking questions can lower your mathematical status in small groups (Ivey, 1997). There is then an asymmetry with respect to authority and questioning for teachers and students. When a teacher asks a question it is an assertion of their knowledge and authority. When a student asks a question it indicates a lack of knowledge and may diminish their authority. A fear of diminished authority, as well as the unknown and unusual nature of the practice, may lie behind the students' anxiety.

There is a diversity of life worlds, identity, and interests

The political nature of classroom communities is brought out more fully when the diversity within the class is considered. The children's life worlds mean that they have different interests in the way teacher-student interactions are conducted. I illustrate these differences with four students by briefly describing aspects of their identity, position, and role within the class and their view of teaching and learning as summarized from interviews and observations. Then I relate these aspects to their preference for how students should answer.

Nikita

Nikita's family recently migrated from Eastern Europe. Conversations and interviews with her reveal a strongly expressed belief in the importance of education that she shares with her family. Nikita believes that the role of teachers is to explain well and that students must listen properly and work hard. She finds mathematics lessons easy and often unchallenging. During teacher questions Nikita rarely volunteers to contribute although she does not find teacher questions a cause of anxiety. As she is confident of her mathematical ability, she does not seek status by answering. Sometimes, rather than appearing to pay attention to the teacher during questioning, she continues to do another task. Nikita accepts that it is part of the teacher's role to ask questions but within her life world such times are a delay to being able to start written exercises. She wants the teacher to exert control so that the ritual of asking questions can be gone through as quickly as possible, and she has no sympathy for others who feel anxiety and want to have time to discuss answers: "I think it's better if someone just ask questions and picks someone."

Susan

Susan is from a White English family. She finds mathematics difficult and she began the year unconfident about her ability. Her relationship with the teacher is central to feeling secure about engaging mathematically. Susan too prefers exercises. Often she will receive individual support from the teacher or another student before starting the exercise, but once she is clear about what she is expected to do she prefers not to be interrupted. Susan is very sociable and interacts with many of the other members of the class. She is a frequent protagonist in argument with boys in the class who, due to their greater numbers and loudness, tend to dominate interactions. Susan wants to be involved in teacher-question interactions. However, teacher questions are a source of anxiety; a wrong answer risks a person being laughed at:[5] "and then you get all, you just get all urggghh [angry and upset sound] and the teacher tells them to stop."

For Susan the fear of embarrassment means that the situations that make her most nervous and those that she finds least helpful to her learning are the same. So when asked to select the situations that were least helpful she responded: "the one when we write it down and the teacher tells us instead of us getting embarrassed when we put up our hand."

In addition, Susan talks favorably about a new strategy that Jill had introduced in which students displayed their answers simultaneously. Susan's attitude indicates the cause of embarrassment is not having to respond publicly in itself, but rather having to respond individually in a public way. In addition these means of answering would mean that there would be more time to arrive at an answer so "you don't get cut off like, where there's brainy people and they like know the answer, and when you go to get them [answers] they just cut you off and tell you the answer when you could have tried."

Lee

Lee is an Afro-Caribbean student. Education is important to his family and this value is a motivating factor at school. Success in tests is primarily about making his family proud. He finds mathematics uninteresting and sometimes difficult. Lessons are an opportunity to socialize. He is part of a group of other boys who spend a good deal of time interacting with each other during mathematics lessons. They communicate by means of "making tunes" through tapping or drumming on tables. The teacher frequently reprimands Lee or spends time trying to get him to work. During teacher questions Lee socializes with whoever he sits near or surveys the classroom, interacting with other boys. Lee does not want to have to respond individually but to have the opportunity to discuss with a peer before answering so that he can be part of a "team."

John

John is a White English boy. He finds the mathematics lessons easy and is often bored. Sometimes he is interested and responds in the way the teacher wants. During other lessons he spends time talking to the person next to him. John is cynical about the world in general and teachers in particular: teachers and the world are waiting to trip him up. He has a world-weary humor that belies his age of eleven years. He does not like the teacher choosing who answers without the students showing that they want to answer.

John: When she asks you for an answer and everybody in class knows it except you, she'll pick you!

MB: How does she know, how does that happen?

John: I don't know why. Some things happen like that. Like if you've got a bunch of money in your hand like a pound and a five p[ence], you'll obviously drop the pound, it just works like that.

Like Susan, the risks of being shown not to know are being laughed at or being embarrassed. John's strategy in mathematics lessons is to choose his level of engagement with activities. Often he does not listen to the teacher's questions. If the teacher picks a student to answer he may be caught out and, given that part of his status in the class is based on generally being able to answer correctly, this social practice is one he dislikes.

Sometimes John participates in teacher-question interactions. However, he is as cynical about the teacher's motivations as the way the fates conspire against him: "If you're like that [raises his hand, leans forward in an eager pose] and you really want to answer it, they look at you and then start looking at everyone else."

John is very concerned about fairness. For him the fairest way is for the students to take turns. Answering questions for John is not primarily about contributing to a process of learning mathematics but about status and identity. There are some questions he really wants to answer; these tend to be the more challenging ones. If students took turns to answer then John would be able to continue with his socializing.

Discussion

Nikita, Susan, Lee and John have four different preferred means of answering questions. Other students responded differently. For example, George and Dave suggested an approach where the teacher allowed every person with their hand up to answer, even if they gave the same answer as the others, before giving an evaluation at the end. Here they take John's concern about fairness further: everyone with an answer should be allowed to speak. Even where students shared a preference, they often did so for very different reasons.

These different responses arise from different life worlds. In each classroom there is an incredible diversity of identity and interests. Nel Noddings (1993) suggests that the mathematics classrooms should be politicized. By this statement she means, in part, transforming classrooms into sites within which students can practice the responsibilities and rights of citizens and so exercise some control over social practices. The diversity of life worlds present in classrooms indicate some of the reasons why this transformation is a difficult project for teachers to undertake. Students have very different ideas of what classrooms should be like and very different needs and desires. Moreover, there is a tendency for all of us to believe that the life world of others is very similar to our own. For example, Susan spoke of "*us* getting embarrassed". My interpretation is that Susan believes she is speaking on behalf of the rest of the class or perhaps her interview group. But though such embarrassment is widespread, it is not universal in the class as a whole and not even shared by all in her interview group.

A recognition of the diversity that exists in a classroom community over a very specific practice such as how questions should be answered helps to reconceptualize the nature of democratic classroom practice. There is no universal ideal that the teacher can implement that will accord with the desires or needs of all. Rather, the teacher has the challenge of finding ways of facilitating dialogue between the different life worlds within the community. From this perspective, democratic practice is concerned with creating the conditions for opening horizons of understanding (Gadamer, 1975) between members of the community.

Diversity in classrooms combined with a scarcity of resources leads to conflict

In the context of teacher questions, the most important resource is the teacher. It is the teacher who decides what is asked and who answers. Status gained by answering correctly is gained through the teacher; thus the teacher's attention is important. A second resource that is very limited is "the first right answer." Once an answer has been given to a closed question then the interaction is over. In Seven B students frequently competed to answer first. This competition could lead to them, mainly boys though not exclusively, "shouting out." Early on in the year Jill noted that for many boys it seemed more important to be the one to answer rather than necessarily to be correct.

While conflict over the teacher's attention may be inevitable in classrooms, it is increased when teacher questions are about being right or wrong. It is not enough for some that they are right; some others must be wrong because "if everyone had the right answer it's not going to be fun."

Particular circumstances within Seven B meant that many of the students' emotional responses about not being chosen were frequently displayed. It is an open question as to how similar the social practices around questioning are in other classrooms, although similar behavior on the part of boys have been reported previously (Zevenbergen, 1999). However, even in classrooms where the surface features are calmer, students will still experience this competitive aspect.[6]

In addition to the competition between individuals, conflict also occurred between groups of students within Seven B, most notably between boys and girls. This conflict is a reflection of and helps to create different life worlds. Some of the boys complained of the girls getting preferential treatment, whereas the girls felt that boys were asked more questions because they would not listen. In Jill's life world she was acting fairly. In terms of what constitutes the community of practice within the classroom all the interactions and conflicts between participants situated within their different life worlds are important.

Communities of political practice

It is my contention that in order to understand how and why participants act in the way they do within classroom communities, the micro-politics of the classroom must be considered. In addition such an analysis helps to clarify the political nature of communities of practice more generally. Classroom communities of practice are political in the following ways:

- They are communities in which individuals or groups hold authority and status that is more or less equally shared and may be contested;
- There is a diversity of life-worlds, identity and interests;
- This diversity, combined with a scarcity of resources, leads to conflict.

This paper has focused on conflict and diversity. There are other more positive features of political life such as agreement, consensus, shared experience and purpose, respect for diversity or mutuality, empathy, and solidarity. These features can also be found within the interactions that constitute the social practice of teacher questioning, although I have not had the space to develop these themes here. I have also not been able to indicate the ways in which the process of opening dialogue with the students about their experiences changed classroom practices.

The students (and teachers) who participate in communities of practice, inhabit life worlds that are formed beyond the individual classroom and bring with them practices from other communities they are part of. These characteristics create boundaries that limit and format the type of practices that occur. However, social practices found within classrooms are also partly the product of the micro-politics of the communities of practice. Recognition of this root of social practices means that there is space to act within the boundaries and gives the possibility of transformation through dialogue.

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[1] This paper was presented at the Third Mathematics Education and Society Conference (MES-3), Helsingør, Denmark, April 2002.

[2] One way of approaching the relationship between the "big" and "small" politics of the mathematics classroom is through a post-structuralist perspective (Hardy, 2000; Hardy & Cotton, 2000; Walkerdine, 1988). Drawing on Foucault, Tansy Hardy points to the way in which relationships of power between individuals are present in classrooms which are arenas in which power and knowledge are inextricably linked (Hardy, 2000). In this paper I avoid referring to "power." I find Foucault's conception of power problematical and not easily related to theories of social practice that have different philosophical roots. A discussion of this relationship is beyond the scope of this paper. However, Foucault's attention to the ways politics is expressed through and about the body (see Danaher, 2000; Foucault, 1995/1975; Simons, 1995) is a productive source for understanding the physical expression of interactions resulting from teacher questions, in particular the issue of surveillance. Again this discussion is beyond the scope of this paper as here I am not largely concerned with observational material. My aim is to indicate ways in which teacher questioning of students is experienced and appears as political rather than to discuss the nature of what it is for something to be political.

[3] All names are pseudonyms.

[4] For those not familiar with London's geography, Maida Vale is a suburb of London. The boys' humor in talking about Maida Vale is intentional and my commentary should not be read as an adult finding humor at the expense of the children but sharing what is definitely the children's joke.

[5] The actual incidence of being laughed at is difficult to determine. I did not witness such incidents during observations of a lesson. However, from within Susan's life world this is what happens: if you are wrong then you are laughed at. This case illustrates the power of considering the life worlds of the students. Regardless of what an observer sees as happening, Susan experiences the classroom as one in which students laugh at each other if they get an answer wrong.

[6] Of course it is not simply classroom communities of practice where micro-political struggles happen over who speaks and when. Such situations occur in meetings, seminars, family dinners, and many more occasions.

Conferences 2002…

ICTM2 International Conference on the Teaching of Mathematics	Crete, Greece	July 1-6
VISIT-ME Vienna International Symposium on Integrating Technology into Mathematics Education	Vienna, Austria	July 10-13
PME Psychology of Mathematics Education	Norwich, UK	July 21-26
MAA The Mathematical Association of America	Burlington, VT	August 1-3
The Humanistic Renaissance in Mathematics Education	Palermo, Italy	September 20-25
PME-NA (see page 4 for more details) Psychology of Mathematics Education-North America	Athens, GA	October 26-29
ICTCM International Conference on the use of Technology in Collegiate Mathematics	Orlando, FL	October 31-November 3

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A Four-Stage Model of Mathematical Learning

Jeff Knisley

Introduction

Research in education and applied psychology has produced a number of insights into how students think and learn, but all too often the resulting impact on actual classroom instruction is uneven and unpredictable (Sabelli & Dede, in press; Schoenfeld, 1999). In response, many in higher education are translating research in education into models of learning specific to their own disciplines (Buriak, McNurlen, & Harper, 1995; Felder, Woods, Stice, & Rugarcia, 2000; Jensen & Wood, 2000). These models in turn are used to reform teaching methods, to transform existing courses, and even to suggest new courses.

Research in mathematics education has been no less productive (Schoenfeld, 2000). This article[1] is in the spirit of those mentioned above, in that I combine personal observations and my interpretation of educational research into a model of mathematical learning. The result of this approach can be used to address issues such as the effective role of a teacher and appropriate uses of technology. That is, the model can be viewed as a tool that teachers can use to guide the development of curricular and instructional reform.

Before presenting this model, however, let me offer this qualifier. In my opinion, good teaching begins with a genuine concern for students and an enthusiasm for the subject. Any benefits derived from this model are in addition to that concern and enthusiasm, for I believe that nothing can ever or should ever replace the invaluable and mutually beneficial teacher-student relationship.

Related Literature

Decades of research in education suggest that students utilize individual learning styles (Bloom, 1956; Felder, 1996; Gardner & Hatch, 1989) and instruction should therefore be multifaceted to accommodate a variety of learning styles (e.g., Bodi, 1990; Dunn & Dunn, 1993; Felder, 1993; Liu & Reed, 1994). Moreover, strategic choices and metacognition are also important in research in mathematics education (Schoenfeld, 2000). Research in applied psychology suggests that problem solving is best accomplished with a strategy-building approach. Studies of individual differences in skill acquisition suggest that the fastest learners are those who develop strategies for concept formation (Eyring, Johnson, & Francis, 1993). Thus, a model of mathematical learning should include strategy building as a learning style.

Some mathematics students employ a common method of learning that might be characterized as the "memorize and associate" method. *Heuristic reasoning* is a thought process in which a set of patterns and their associated actions are memorized, so that when a new concept is introduced, the closest pattern determines the action taken (Pearl, 1984). Unfortunately, the criteria used to determine closeness are often inappropriate and frequently lead to incorrect results. For example, if a student incorrectly reduces the expression $\sqrt{x^4 + 4x^2}$ to the expression $x^2 + 2x$, then that student likely used visual criteria to determine that the closest pattern was the root of a given power. In mathematics, heuristic reasoning may be a sign of knowledge with little conceptual understanding, a short circuit in learning that often prevents critical thinking. Using heuristic reasoning repeatedly is not likely to build a strong foundation for making sense of mathematics.

I believe that the learning model most applicable to learning mathematics is *Kolb's model of experiential learning* or *Kolb's model*, for short (Evans, Forney, & Guido-DiBrito, 1998). This belief grows out of my experience teaching mathematics, but Kolb's model has also been used extensively to evaluate and enhance teaching in engineering (Jensen & Wood, 2000; Pavan, 1998; Stice, 1987; Terry & Harb, 1993).

In Kolb's model, a student's learning style is determined by two factors—whether the student prefers the concrete to the abstract, and whether the student prefers active experimentation to reflective observation. These preferences result in a classification scheme with four learning styles (Felder, 1993; Hartman, 1995):

- Concrete, reflective: Those who build on previous experience.
- Concrete, active: Those who learn by trial and error.
- Abstract, reflective: Those who learn from detailed explanations.
- Abstract, active: Those who learn by developing individual strategies.

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These learning styles are not absolute, and all learners, regardless of preference, can function in all four styles when necessary (Kolb, 1984; Sharp, 1998). Indeed, in the *Kolb learning cycle,* each style is considered a stage of learning and students learn by cycling through each of the four stages (Harb, Durrant, & Terry, 1993; Kolb, 1984; Pavan, 1998). For example, the cycle begins with the student's personal involvement through concrete experience; next, the student reflects on this experience, looking for meaning; then the student applies this meaning to form a logical conclusion; finally, the student experiments with similar problems, which result in new concrete experiences. From here, the learning cycle begins again (Hartman, 1995).

Kolb Learning in a Mathematical Context

Kolb's learning styles can be interpreted as mathematical learning styles. For example, "concrete, reflective" learners may well be those students who tend to use previous knowledge to construct allegories[2] of new ideas. In mathematics courses, these learners may approach problems by trying to mimic an example in the textbook. Based on several years of observation, experimentation, and student interaction, I have interpreted Kolb's other three learning styles in a mathematical context:

- *Allegorizers:* These students consider new ideas to be reformulations of known ideas. They address problems by attempting to apply known techniques in an ad-hoc fashion.
- *Integrators:* These students rely heavily on comparisons of new ideas to known ideas. They address problems by relying on their "common sense" insights—i.e., by comparing the problem to problems they can solve.
- *Analyzers:* These students desire logical explanations and algorithms. They solve problems with a logical, step-by-step progression that begins with the initial assumptions and concludes with the solution.
- *Synthesizers:* These students see concepts as tools for constructing new ideas and approaches. They solve problems by developing individual strategies and new allegories.

The table in Figure 1 shows the correspondence between Kolb's learning styles and my interpretation in a mathematical context:

KOLB'S LEARNING STYLES	EQUIVALENT MATHEMATICAL STYLE
Concrete, Reflexive	Allegorizer
Concrete, Active	Integrator
Abstract, Reflective	Analyzer
Abstract, Active	Synthesizer

Figure 1. Kolb's Learning Styles in a Mathematical Context.

Moreover, I have not only observed that students are capable of functioning in all four styles, but that the preferred learning style of a student may vary from topic to topic. For example, students with a preference for synthesizing with respect to one topic may change to a preference of integration for another topic, and vice versa. In addition, when a student's learning style does not facilitate successful problem solving, I have observed that the student often resorts to heuristic reasoning.

These observations have led me to hypothesize that a student's preferred learning style for a given concept may indicate how well that student understands that concept. That is, a student's learning style preference may be a function both of the content and the level of understanding of the material. The existence of at least four different *styles* of learning may be indicative of at least four different *stages* of understanding of a mathematical concept, which again is in agreement with one of Kolb's original observations (Smith & Kolb, 1986). I believe this relationship can be used to improve instruction—i.e., a teacher's identification of how well students understand a topic can be used to design instruction so that it best addresses students at that level of understanding.

Stages of Mathematical Learning

There are models with more than four learning styles, and there may be models with more than four stages of mathematical learning. Furthermore, a given student may prefer a learning style for some reason other than level of understanding. However, I believe that if a large number of students in a given classroom prefer a particular learning style for a given concept, then that may indicate how well that group of students understands that topic (Felder, 1989, 1990, 1996). In fact, my experience in teaching mathematics suggests that it is useful to view each learner as progressing through the following four distinct stages of learning when acquiring a new concept.

- *Allegorization:* A new concept is described figuratively in a familiar context in terms of known concepts. At this stage, learners are not yet able to distinguish the new concept from known concepts.
- *Integration:* Comparison, measurement, and exploration are used to distinguish the new concept from known concepts. At this stage, learners realize a concept is new, but do not know how it relates to what is already known.
- *Analysis:* The new concept becomes part of the existing knowledge base. At this stage, learners can relate the new concept to known concepts, but they lack the information needed to establish the concept's unique character.

- *Synthesis:* The new concept acquires its own unique identity and thus becomes a tool for strategy development and further allegorization. At this stage, learners have mastered the new concept and can use it to solve problems, develop strategies (i.e., new theory), and create allegories.

That is, a student may prefer allegorization as a learning style only until he realizes that the idea they have been exposed to is a new one, after which that same student may prefer the comparisons and explorations that characterize integration. Similarly, once a student understands how the new concept compares to known concept, then she may desire to know all there is to know about the concept, and having done so, she may ultimately desire the mastery of the topic implied by a preference for synthesis. Thus experiencing different learning stages may impact the learning style of the student

The Importance of Allegories

Given that a student's preferred style may be due in part to a student's current level of understanding of a concept, the four stage model described in the previous section suggests that learning new concepts may fruitfully begin with allegory development. That is, a figurative description of a new concept in a familiar context may be a useful intuitive introduction to a new idea and should precede any attempts to compare and contrast the new idea to known ideas. Indeed, a student with no allegorical description of a concept may resort to a "memorize and associate" style of learning.

To illustrate the importance of allegory development, let us consider what might transpire if I were to teach a group of students the game of chess without the use of allegories. I would begin by presenting an 8 by 8 grid in which players 1 and 2 receive tokens labeled A, B, C, D, E, and F arranged as shown in Figure 2.

B1	C1	D1	E1	F1	D1	C1	B1
A1	A1	A1	A1	A1	A1	A1	A1
A2	A2	A2	A2	A2	A2	A2	A2
B2	C2	D2	E2	F2	D2	C2	B2

Figure 2. Chess without Allegories.

I would then explain that each type of token has a variety of acceptable moves—e.g., the "B" tokens can move vertically or horizontally but must stop when encountering another token, whereas "C" tokens have four possible L-shaped moves and need not stop if other tokens are in those paths. I would conclude my explanations by stating that the goal of the game is to immobilize the other player's "F" token. In response, students would likely memorize valid moves for each token, and then would memorize when to make those moves—a way of "playing" that does not seem like much fun.

In contrast, I believe people learn and enjoy chess *because* the game pieces themselves are allegories within the context of medieval military figures. For example, pawns are numerous but have limited abilities, knights can "leap over objects," and queens have unlimited power. Capturing the king is the allegory for winning the game. In fact, a vast array of video and board games owe their popularity to their allegories of real-life people, places, and events.

Thus, when I teach a course such as calculus or statistics, I try to develop an allegorical introduction to each major concept. To do so, I begin by identifying a context that is appropriate for a given class at a given time. For example, most of my students enter calculus with decent arithmetic skills; a limited background in algebra; and a mostly underdeveloped understanding of geometry, trigonometry, and functions. Correspondingly, I usually introduce calculus by using algebra and arithmetic to explore tangent lines to polynomial curves. In contrast, many calculus textbooks begin with limits of functions, including transcendental functions. I would argue that such an introduction does not lend itself to allegorical description and that the result is that calculus students are well entrenched in heuristic reasoning by the time they take the first test.

As another example, consider that when students hear the word "probability," they most likely think of rolling dice and flipping coins. If so, then random walks constitute a natural allegory for introducing nearly all of the primary ideas in statistics and probability. However, a course in probability and statistics often introduces normal distributions, statistical tests, expected value, and standard deviation as if they are intuitively obvious. My experience is that even when students make high grades in a statistics course, statistical concepts remain mysterious to them.

Components of Integration

Once a concept has been introduced allegorically, it can be integrated into the existing knowledge base. I believe that this process of integration begins with a *definition*, since a definition assigns a label to a new concept and places it within a mathematical setting.

Once defined, the concept can be compared and contrasted with known concepts.

Visualization, experimentation, and exploration can play key roles in integration. Indeed, visual comparisons can be very powerful, and explorations and experiments are ways of comparing new phenomena to well-studied, well-understood phenomena. As a result, the use of technology is often desirable at this point as a visualization tool

For example, suppose that a certain class of students has a good grasp of linear functions and suppose that exponential growth has been allegorized and defined. It is at this point that students may best be served by comparisons of the new phenomenon of exponential growth to the known phenomenon of linear growth. Indeed, suppose that students are told that there are two options for receiving a monetary prize—either $1000 a month for 60 months or the total that results from an investment of $100 at 20% interest each month for 60 months. Visual comparison of these options reveals the differences and similarities between exponential and linear growth (see Figure 3). In particular, exponential growth appears to be almost linear to begin with, and thus for the first few months Option 1 will have a greater value. However, as time passes the exponential overtakes and grows increasingly faster than the linear option, so that after 60 months, Option 1 is worth $60,000 while Option 2 is worth $4,695,626.

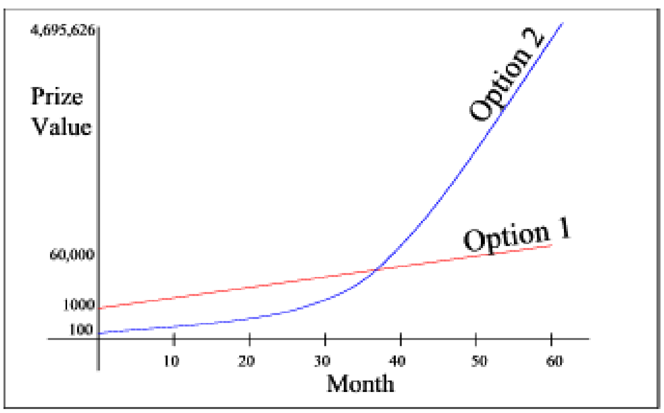

Figure 3. Visual Comparison of Linear and Exponential Growth.

Other comparisons that may be appropriate at this point include comparing an exponential to polynomials of increasing degree or comparing a sine wave to an exponentially damped sine wave. In my opinion, comparisons such as these are of no value before a student realizes that exponential growth is a new type of growth they have not yet imagined in their context of algebraic functions. Moreover, presenting a comparison such as Figure 3 to a group of students who have spent some time concentrating on the properties of the exponential may lead them to wish audibly that they had seen Figure 3 before studying all those unmotivated properties. That is, comparing new ideas to known ideas seems to me to be most natural and most beneficial in the second stage of learning.

Analysis

Once a student has experienced an allegorical introduction to a new concept and has compared the new concept to known concepts, he is ready to consider the new concept independent of other ideas. Indeed, at this stage, the new concept takes on its own character, and the student's desire is to learn as much as possible about that character. Learners in the analysis stage want to know the history of the concept, the techniques for using it, and the explanations of its different attributes. Furthermore, they want information about the relationship of the new concept to known concepts that goes beyond comparisons, such as the sphere of influence of the new concept within their existing knowledge base.

As a result, learners in the analysis stage desire a great deal of information in a short period of time. Thus it seems appropriate to lecture to a group of such learners. Unfortunately, many of us who teach mathematics too often assume that all of our students are at the analysis stage for every concept, which means that we deliver massive amounts of information to students who have not even realized that they are encountering a new idea. This phenomenon appears to occur for the limit concept in calculus. Studies have shown that few students complete a calculus course with any meaningful understanding of limits (Szydlik, 2000). Instead, most students resort to heuristics to survive the initial exposure to the limit process.

Synthesis

Finally, the synthesis stage involves mastery of the topic, in that the new concept becomes a tool the student can use to develop individual strategies for solving problems. For example, even though games often depend heavily on allegories, some would argue that the fun part of a game is analyzing it *and* developing new strategies for winning. Indeed, most people would like to reach the point in a game where they are in control—that is, the point where they are synthesizing their own strategies and then using those strategies to develop their own allegories of new concepts.

However, synthesis is a creative act, and not all students will be able to act as synthesizers with a given concept within the same period of time. The cycle of learning may break down at this point due to an inability to use the concept under study to generate allegorical descriptions of a subsequent concept. Consequently, learning mathematics may not be feasible for most students without the assistance of a teacher.

The Role of the Teacher

The four stages of mathematical learning cannot be reduced to an automated process with four regimented steps. Appropriate allegories should be based on a student's previous experiences, and consequently new allegories must be continually developed. Some concepts require more allegorization, integration, and analysis than others, and it may not be a judicious use of time to ask students to synthesize their own allegories for new ideas.

As a result, there must be an intermediary—i.e., a teacher—who guides the development of allegories for the students, who determines how allegorization, integration, and analysis should be used in presenting a concept, and who prompts students to synthesize and think critically about each concept.

Indeed, it has been suggested that the ideal classroom would include each of the four processes in the Kolb cycle (Hartman, 1995; McCarthy, 1986). That is, full comprehension requires learning activities fitting each stage of learning (Jensen & Wood, 2000). McCarthy has identified four roles for the teacher based on the Kolb learning cycle—evaluating, motivating, teaching, and coaching.

Likewise, the four stages of mathematical learning described above imply at least four different roles for the teacher of mathematics.

1. *Allegorization:* Teacher is a storyteller.
2. *Integration:* Teacher is a guide and motivator.
3. *Analysis:* Teacher is a source of information.
4. *Synthesis:* Teacher is a coach.

In the stages of allegorization and analysis, the role of the instructor is one of active leadership, while in the stages of integration and synthesis, the instructor is a mentor, guide, and motivator who emphasizes active learning, exploration, and expressions of creativity.

I will explore each of these roles in turn. When a teacher first introduces a concept to a group of students, the teacher may act as a storyteller to meet the students' need for allegorization. That is, students need a teacher to provide intuitive introductions to new ideas in familiar contexts—historical, arithmetic, scientific, or otherwise. For example, even though I teach college students, I keep a set of measuring cups in my office as an allegory for arithmetic of fractions. Usually, using the measuring cups to demonstrate that $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ is more than sufficient to motivate the idea of a common denominator.

Students who have realized that a new idea is being considered need to compare and contrast that new idea to known ideas. Thus, a teacher may find it fruitful to define the new idea in a way that allows it to be differentiated from known ideas, and then may engage students in focused exploration that will reinforce and clarify the comparisons of the newly defined concept to previously defined ideas. Among these comparisons may be visualizations and numerical experiments with a predicted outcome that must be prepared in advance.

Students who understand the nature of a new concept are ready for someone to provide a great deal of information about the concept in a short period of time. Thus, students who are in the analysis stage may benefit from a teacher who knows the subject in great depth and detail. In addition, students in this stage may benefit from a teacher who provides a number of different sources for information about the idea.

Students who are at the stage of synthesis still need a teacher to advise and direct them. That is, a teacher in the role of a coach may foster the growth of these learners by helping them to develop discipline and structure in their creative activities. Personally, I believe that many of the students who feel bored or even stifled in our educational system are students with great potential who are waiting for someone to offer them a different direction. Thus, teachers need to foster in all students the realization that doing mathematics is a creative activity and that such creativity is both enjoyable and rewarding.

Conclusion

Educational research, applied psychology, and research in mathematics education have produced a great many insights and potential improvements to mathematical instruction. However, as has been realized in other fields, it is important that teachers translate the results of that research into a form appropriate for use in the classroom. Sabelli and Dede (in press) use the phrase "Scholarship of Practice" to describe this idea.

The four stages of mathematical learning presented in this article speak to this purpose. Educational researchers have demonstrated the importance of multiple learning styles. Applied psychologists have established the importance of strategy-building and stages of skill development. Mathematical researchers have identified many areas where mathematical instruction can and needs to be improved.

I have simply combined these ideas into a working model that describes what students may experience in mathematics courses. The model suggests that concepts need to be allegorized first, integrated next, analyzed third, and synthesized last. It also implies that teachers should play many different roles in the classroom to meet students' needs in the different learning stages—for example, adopting the role of storyteller during the allegory stage and acting as a coach at the synthesis stage.

This model has become an invaluable tool in my own teaching. It allows me to diagnose student needs quickly and effectively; it helps me budget my time and my use of technology; and it increases my students' confidence in my ability to lead them to success in the course. I hope it will be of equal value to my fellow educators in the mathematics and mathematics education professions.

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[2] An allegory is a figurative description of an unknown idea in a familiar context.

Mathematicians' Religious Affiliations and Professional Practices: The Case of Joseph

Anderson Norton III

The mathematics education community fosters discourse on a wide variety of personal and social factors influencing mathematical development in the individual and in the mathematics community. Some authors have focused on the issues of race and gender in mathematical learning (e.g. Moody, 1998; Fennema, 1990). Others have focused on the issue of social norms in classroom mathematical development (e.g. Cobb, Wood & Yackel, 1991; Lampert, 1990). Still others have tried to reveal the long history of insights that have determined the fate of mathematical development (e.g. Kline, 1980). Such work reveals the overlap between our lives as humans and our lives as teachers, researchers, and students of mathematics.

Throughout all of the discussion of humanizing mathematics, one facet of our lives is blatantly omitted: religion. Religion (and not politics) remains a taboo topic for us as researchers. It seems that separation of church and state has extended to research in mathematics education. I searched several library and Internet resources looking for studies on the relationship between religion and mathematics learning and teaching; I found none. What I did find were a couple of reports on policies of segregation for religion and science in our schools, and biographies that included theological confessions of historical figures in mathematics.

In "The Science and Religion Wars," Singham reported that 40% of scientists believe in a deity (2000, p. 430). However, he argued that faith in science might crumble under a God who intervenes in the world: "If the scientific community concedes even one miraculous event, then how can it credibly contest the view that the world (and all its fossilized relics) was created in one instant just 6,000 years ago?" (p. 428). Likewise, Warren Nord found that "as it is practiced, science assumes God is irrelevant to understanding nature" (1999, p. 29). The National Academy of Sciences seems to condone such practice: "Religion and science are separate and mutually exclusive realms of thought whose presentation in the same context leads to a misunderstanding of both scientific theory and religious belief" (p. 29). These statements might indicate there are no implications of religion to be found among scientists or mathematicians. Given the powerful roles that mathematics and religion can play in a person's life, I find this conclusion hard to believe. Nord suggests one possible resolution for religious scientists by noting that evolution and other scientifically defined processes may just be "God's way of doing things" (p. 30). Joseph's case introduces another resolution. By way of his story, the present study investigates the ways in which religion might influence mathematical research and teaching, views of mathematics, and one's decision to study mathematics in the first place.

Mathematics educators have looked to the practices of professional mathematicians in order to build metaphors for classroom learning (Cobb, Wood, & Yackel, 1991; Lampert, 1990; Nickson, 1992) under the assumption that understanding their motivations, perspectives and methods leads to a better understanding of mathematics itself and our practices as mathematics educators. I began my study of mathematics and religion under the same assumption. I feel that we can learn a great deal about the religious facet of mathematical development by examining professional mathematicians who hold strong religious convictions. With this assumption in mind, the purpose of my study was to investigate the implications of particular religious affiliations in the lives of professional mathematicians. How do strong religious convictions influence their mathematical practices (research and teaching) and their views of mathematics? Though my larger study includes Buddhist, Christian, and Jewish participants, I focus on the Jewish participant in this paper.

While the biographies I found in my initial search offer perspectives on personal relationships between mathematicians' views of mathematics and their religions, references to such perspectives are often spotty, and there is little or no mention of practice. Since these mathematicians are all dead (some for decades or centuries), the biographies do not engage us in the present state of mathematics. Therefore, these biographies do little to describe current mathematical practices, and they do not answer my research question. Instead, I use them as backdrops to set the stage for Joseph's story. I will reference these histories in building my discussion and conclusions.

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The Histories

In *Kepler's Tübingen: Stimulus to a Theological Mathematics*, Charlotte Methuen (1998) identified four historical relationships between mathematics and religion: conflict, independence, dialogue, and integration. She recounted the life and theory of the 16th century philosopher, Philip Melanchthon. Melanchthon clearly fell into the last category, claiming, "the study of mathematics offers a vehicle by which the human mind may transcend its restrictions and reach God" (p. 83). This rather bold statement relies on the certainty of mathematics and God's order of nature.

On the other hand, the 20th century mathematical logician, Bertrand Russell, relied solely on the certainty of mathematics. "For a period of his life his attitude towards mathematics made up a great part of his personal religion" (Anderson, 1994, p. 2). He had given up on trying to find truth in his Christian religion at a young age. He needed a new religion and sensed that he could find truth in mathematics. This led to a study of the foundations of mathematics and an attempt to ground it in logic - a project that culminated in his publication of *The Principles of Mathematics.* By age thirty-eight, however, he was discouraged by the problems with mathematical foundations (revealed in part by his own work) and was led to give up on the certainty of mathematics as well.

In *The Man Who Loved Only Numbers*, Paul Hoffman (1998) wrote about a brilliant mathematician who found truth in mathematics. Paul Erdös was a Hungarian man who traveled the world to work with other mathematicians on proving theorems, until he died a few years ago at age 83. Hoffman described him as "a mathematical monk… uncovering mathematical truth" (1998, p. 25). Erdös envisioned a God (known as SF, or the Supreme Fascist) who held a book of mathematical truths; "You don't have to believe in God, but you should believe in the book," said Erdös (p. 26). He cursed SF (in whom he himself hardly believed) for keeping this book of truths from him.

Concerning the existence of "the book," Hoffman (1998) claimed "if you believe in God, the answer is obvious. Mathematical truths are there in the SF's mind and you just rediscover them" (p. 26). To illustrate the position, Hoffman offered the story of Ramanujan, perhaps the brightest mathematician ever, who received mathematical knowledge in his dreams from the goddess Namagiri. Ramanujan believed in the truth of mathematics, but as a Hindu, he also believed in God: "an equation for me has no meaning unless it expresses a thought of God" (p. 85).

Einstein, on the other hand, did not believe in a personal god. Instead, in *The World as I See It* he wrote about a "cosmic religious feeling" (1990, I, p. 26). He claimed that Buddhism had a strong element of this feeling. Far from believing that science and religion were at odds with one another, he claimed, "in this materialistic age of ours the serious scientific workers are the only profoundly religious people" (p. 28) because they are able to think abstractly and universally. In *Out of My Later Years*, Einstein pointed out that "the realms of religion and science are clearly marked off from each other" in that they answer different questions (1990, II, p. 26). Still, he proclaimed, "science without religion is lame; religion without science is blind" (p. 26).

Methods

In order to study the implications of religious affiliations in the lives of professional mathematicians, I conducted interviews with three university mathematics professors. With the help of two professors in the mathematics department of a large, southern university, I identified three religious groups representing the diversity of religious beliefs in their mathematics department: Jewish, Christian and Buddhist. I knew the Jewish participant (the one described in this paper) better than the others because I have talked with him on several occasions at mathematical meetings and social gatherings with his department. As a Catholic, I was also somewhat familiar with his religious doctrine.

I conducted a single one-hour interview with each participant using these questions:

1. Describe your beliefs concerning religion.
2. How do these beliefs affect your lifestyle?
3. Tell me about your decision to study mathematics.
4. Tell me about your role as a professional mathematician.
5. Do you see any relationship between your professional practice and your religious beliefs?

For background information, I collected additional data from archival sources including vitas of the participants and a booklet describing the faculty of their department. I used memoing to develop codes from the data and then grouped codes into categories to identify concepts. I constructed narratives from the concepts, but I wanted to include something additional to capture the words and phrases of my participants. So I incorporated poetic transcription (Glesne, 1999, pp. 183-187), restructuring words from the transcripts into poems. In forming the stanzas, I was careful to stay close to my interpretations of their meaning. While I used only the literal phrases and words of the participants in this section, their order and concatenation may be very different from the literal transcriptions. I hope that the end result gives the flavor of the participants' voice and language that is missing from the narratives.

Joseph's Story

Background.

Joseph is a Jewish man of about fifty-five years. He was raised in a conservative Jewish family; his mother was especially conservative in observing Jewish laws. His beliefs are mostly orthodox, which means that he believes that the Jewish Bible (the Torah) is the word of God handed down to Moses. He also recognizes the laws passed down through oral tradition and later recorded in the Talmud.

Study, both scriptural and worldly, is very important to Joseph. He studies the Talmud with a friend in the philosophy department. For his studies in mathematics, he received a doctorate from the University of Michigan. He is particularly interested in functional analysis, a subject in which he has a long list of publications.

Teaching is also important to Joseph. He has taught 68 different courses in mathematics at his university and has gained much respect in his 28 years of teaching there. In fact, he recently received a prestigious university award for his teaching. His students celebrate his patience, humor and dedication in the classroom. His services to students extend to various other activities as well: judging science fairs, sitting on the Academic Dishonesty Panel, and serving on many graduate student committees.

Learn, Obey, Teach.

Orthodox Jews believe that the Bible is the written word of God handed by Him to Moses on Mount Sinai. Therefore, "[their] primary responsibility is to learn, obey, and teach" the commandments written there, as well as those passed down through oral tradition and recorded in the Talmud. As part of his responsibility to learn, Joseph emphasized the importance of "study for its own sake." Studying the Torah and the Talmud is a way of "showing love for God," but there is also a religious value in studying other things, such as mathematics. He notes that mathematics too requires a respect for study. This value establishes one of many relations between Joseph's religion and his profession. "Doing what you can" and "the value of teaching others" also appear across domains for Joseph, as do many of his beliefs and practices. Though the relations are clear, it is not so evident that aspects from one domain influence the other. As Joseph put it, "I don't know whether that is [the] influence of my religious experience or just the way that I am."

The Talmud includes commentaries that explain the logic of some laws and clarify the meanings of some words. When Joseph is reading the laws in the Talmud, he first tries to figure out their meaning without the aid of the commentary. Often he is unsuccessful and has to look for hints in the commentary before returning to the law. However, this hermeneutic process of text interpretation helps him to understand the law better, and it is the same process he uses in reading mathematical proofs:

> I'll try to prove it myself, and then when I get stuck, I'll look at their proof and try to find the idea that I am missing. And then I'll try to do it myself—that process. Of course, that takes a very long time, and I can't do it for many papers, but whatever I do succeed in doing along the way really becomes mine. And I guess that's an influence of studying the Talmud. You read the text and then you try to figure out the reasoning for yourself.

Joseph feels that when he tries to figure things out for himself, he understands them better. He doesn't like to take things for granted, though he admits that in Judaism there are things that he must accept on faith, "the purest form of obeying God." Human understanding is limited; only God has perfect knowledge. To think that people can attain ultimate truth through mathematics is nothing more than "human chauvinism—the glorification of reason." Although logicians have tried to establish the certainty of mathematics on logical foundations, Joseph claims that "[mathematicians] are scared of them [logicians]." Rather than worry about foundations, he focuses on doing his job. "Just because you can't do something completely, doesn't mean you can't make some progress. That's certainly built into my religion—that attitude."

Joseph's practice of studying the Talmud and mathematics texts demonstrates another relationship between his religion and his profession. Passed down through oral tradition and later written in concise form, the Talmud is a center of ongoing discussion. Many of the laws are explained in a theorem-proof fashion in which reasons are given and then their necessity is justified logically. People voice different opinions as to the meanings of laws and even individual words written there, and others raise objections to those opinions. In this way, the laws are sort of flexible and debatable. The objective of interpretation is to try to establish a consistent opinion from which to understand the laws.

This objective is similar to debate in mathematics. While the results of mathematics are published in "finely polished texts," the real, intuitive work of mathematicians that lies behind the text is much less defined and is open to argument. Like the Talmud, mathematical theorems and proofs should be concise, and the meanings of the words it uses are crucial to understanding. Disagreement over the meaning of just one word in either domain causes confusion. Still, as Joseph sees it, “you never completely understand a definition.” This conclusion highlights yet another common aspect of Joseph’s religious and mathematical studies.

Joseph’s interview responses are wrought with examples, which seem equally important to him in understanding Judaism and mathematics. Looking at examples in the Talmud helps him to refine the meanings of religious laws. Looking at examples in mathematics helps him to establish the boundaries of theorems and definitions. In fact, Joseph like the way Halmos put things in saying that “theorems are the afterthoughts of examples.”

As an Orthodox Jew, Joseph tries to do the things God wants for him to do and apply the Jewish law in his life each day. He tries to understand the reasons for God’s law in order to understand its application in his life, rather than to establish why God made the law. “Our primary religious obligation is to obey the commandments that God gave us… and it's impossible to conjecture what God is like.” This attitude applies to other philosophical speculations as well, such as the ones about the afterlife. There is certainly a belief in an afterlife in Judaism, but “there is little conjecture as to what the world to come will be like.”

Joseph’s attitude of doing his job with little concern for philosophical questions carries over to his mathematical practice. He thinks it is important for students and researchers alike to make as much progress as they can on mathematical problems. It’s hard to say how much this attitude reflects Joseph’s religious beliefs and how much of it is just part of his personality, but the idea of doing what you can is certainly an important value in both his religious and mathematical practice.

One can make a stronger argument that these relationships are actually influences of his religion upon his profession. Joseph perceives that “there are a disparate number of religious Jewish mathematicians” because of the “similarity of the activities and because there is no potential conflict with mathematics as in other sciences because mathematics is self-contained and built on our own axioms.” Thus, Joseph identified logical grounds for his affinity for mathematics and his application of religious practices to that domain.

Aside from study, there is another religious value that is central to his profession. Joseph feels that “the value of teaching is ingrained [in me]… I'm sure it influences the way I look on the profession.” When it comes to teaching, he seems keenly aware of the influences. One could even say that Joseph’s dedication to teaching mathematics is religious.

Many aspects of Joseph’s religious *practice* affect his teaching as well. For example, he finds it important to use a lot of examples in his classes. While students often view examples as models of solutions to a class of problems, Joseph uses them to help students understand concepts. He also thinks that students should get used to studying a good mathematical text, in much the same way he studies the Talmud. “[My teaching] is influenced by studying the Talmud...You read the text, but most of the time that you spend is trying to understand the logic of it and reconcile the different opinions.” In fact, he feels that part of his responsibility to the students is to help them learn to read the text. Understanding the logic is at least as important to studying a mathematical text as it is to studying the Talmud. After all, “the rules of logic are pure and precise in mathematics as nowhere else.”

Joseph doesn’t like to lecture. He feels that it is more beneficial to the students if he answers their questions and shares perspectives with them. This seems to be related to his attitude toward authority; the students should be doing, rather than blindly following. They should be trying their own ideas and developing mathematics on their own as much as they can. Mathematics professors should help students to act more like researchers. “The idea is actually to get [students] to do some research themselves.” In fact, Joseph talked about a grant received by his department to do just that.

In Joseph's case, we can be certain of many relationships that exist between his religion and his profession. Joseph himself noted the disproportionate number of religious Jews in his field. In the case of his religious values for study and teaching others, the influence seems clear. In the case of his religious practice, such as his method of studying the Talmud, the influence may be mutual or the result of a third cause (“just the way I am”). Still, influential or not, the many relationships described here go far beyond coincidence.

How are we going to deal with the fact that we're all going to sin?
God knows we're not perfect, but that does not release you
From the responsibility of doing what you can. No matter where you are,
There is a right thing to do at this point.

God is the perfect everything. Showing love for God is primary
Religious obligation—to learn, obey, and teach the commandments He gave us.
A fanatic observes one more than you; A heretic observes one less.
But, doing the commandment that you don't understand is the purest form of obeying.

Just because you can't do it completely
Doesn't mean that you can't make some progress.
That's certainly built into my religion—that attitude—
Is very much like doing mathematics. It's not a spectator sport.

Mathematics is self-contained and built on our own axioms.
Not many of us are going to question the Law of the Excluded Middle.

Yet we all use the Axiom of Choice and we don't apologize for it.

We won't conjecture as to why is this really a good axiom.
It's the logicians who do this, and we're scared of them!
The same thing motivated the Greeks. They wanted something
To believe in: human chauvinism and the glorification of reason.

Uncertainty Principle, Incompleteness Theorem:
"There are limits to what we can know and what we can understand."
There's no point in conjecturing as to the afterlife. That doesn't help you
Do a better job of doing what He wants.

The Talmud is a commentary on the living portion of law.
In concise form, studying that is a lot like studying mathematics.
It was necessary to list all those reasons a person should not go into a ruin.
The hypotheses really were necessary—indispensable commentary.

I don't like taking things on authority. I can't read another person's proof.
We try to figure it out by ourselves. I try to prove it myself.
And then when we can't do it, and then when I get stuck,
We look at the commentary. I'll look at the author's proof.

Then it really becomes mine. Well, that's just the way I am.
You read the text and try to figure out the reasoning for yourself.
The idea of study for its own sake—that's something that is ingrained
I'm sure it influences the way I look on the mathematics profession.

Language poses some difficulties. You never completely understand
A definition: a word that is actually showing for you to learn this extra lesson.
There's nothing that doesn't have a purpose and I guess
That's really similar to my attitude towards learning mathematics.

You don't know what's true. You have to start looking at examples, and examples
Show the boundaries of a theorem. People have this attitude toward mathematics
That it's very well defined. Intuition goes on behind it, without which
The whole enterprise is meaningless.

The value of teaching, that's built in. I don't like to lecture.
Looking at examples all the time. For example,
20 ideas in 5 minutes, one will probably have some seed. There's nothing wrong
With wild conjectures and making mistakes—this cyclical idea.

Working Toward Reconciliation

I recall one of my undergraduate mathematics professors telling me "mathematics is the only truth with the possible exception of theology." The histories recounted here along with Joseph's story offer illustrations of ways that mathematical truth and theological truth might co-exist. Hoffman (1998) questioned the existence of a mathematical bible ("the book") and presented the stories of Erdös and Ramanujan to exemplify two possible positions. While Ramanujan believed in a god who holds the book, Erdös believed in the book without holding a god. Melanchthon's story provides us with a Christian perspective similar to that of Ramanujan's Hindu perspective. Russell seemed to recede from a position similar to Erdös' into a third position that neither accepted a God nor the book. Joseph's story offers the fourth possible resolution—a god without a book—refuting Hoffman's claim that a belief in God necessarily implies a belief in the book.

Joseph's resolution depends upon the limits of human understanding and the boundaries between our mathematical knowledge as humans and God's Truth that cannot be known to us. Whereas Melanchthon pursued mathematical understanding as a means to understanding God, Joseph pursues mathematical knowledge as a meritorious activity within our restricted domain of understanding. He does not make any claims about the universal truth of mathematics. He warns against such claims as a product of the "glorification of reason." Instead, he views mathematics as a closed system, "built on its own axioms." In fact, Joseph says that the closure of mathematics distinguishes mathematics from other sciences that might conflict with his religious beliefs. This orientation fits Methuen's (1998) idea of an independent stance.

Though Joseph draws a distinction between his mathematical understanding and his understanding of God (thus making the two independent), it is important that he finds religious meaning for his activities as a mathematician. He believes it is important that he does what he can. This Jewish belief, along with the value for study, may have influenced his decision to enter his profession. In fact, these beliefs are the foundations for Joseph's Jewish practices, which he identified in explanation for the disproportionate number of Jewish mathematicians.

Whether or not his religion influenced his decision to become a professional mathematician, Joseph's religious values and practices certainly fit his profession. As noted earlier, Joseph's practice of studying the Talmud carries over to his mathematical practice. Looking at examples and struggling with definitions is important to both domains. In reading mathematical proofs, he tries to reproduce as much of the proof as he can on his own before looking at the original proof for hints. This approach is the same one he takes to studying the Talmud. In both activities he senses the responsibility to "do what you can." In addition, Joseph believes there is a religious value of teaching, and his religious practices of studying texts and looking at examples extends to that aspect of his profession. He feels that he should provide a good text for his students to study at home, while spending class time providing examples and perspectives on the reading.

Through his story, Joseph teaches the mathematics education community something as well. To understand the mathematics profession deeply, we must reconcile it with our deepest held beliefs and values. For those without strong religious convictions, these beliefs and values may stem from a philosophy of life (as they did for Russell and Erdös.) Without this reconciliation, our profession lacks meaning. For teachers, this absence of meaning would be disastrous. How are we to teach children mathematics when we cannot answer for ourselves what mathematics is and why it is important? Worse yet, how can we profess mathematics when there is unresolved conflict between our own mathematical beliefs and religious convictions? Though our answers to these questions will vary, each mathematician and mathematics educator must develop a philosophy of mathematics that can coexist with her philosophy of life. In Joseph's case, his mathematical beliefs might be considered independent of his religious beliefs, but there is harmony between his religious values and his professional practice, and his religion helps to define his professional practice as meritorious activity.

Teaching is also a meritorious activity for Joseph. Just as the value of religious study extends to his study of mathematics, the value of teaching the Talmud seems to extend to his mathematical teaching. For some mathematics teachers, the value of teaching may encompass the entire value of their profession. However, it is Joseph's value of mathematical study combined with his value of teaching that enables him to teach mathematics passionately. If we want our students to act as mathematicians, we need to convey the significance of the subject through our teaching. We need to know at a philosophical level what mathematics is and why we are teaching it.

As a final note, Joseph's story might awaken us to particular approaches our students take to mathematical study. We have seen that Joseph's learning style is informed by his religious practice. We might expect similar influences for students who engage in ritual religious study. After all, many students have developed their styles to studying religious texts, such as the Koran or the Bible, over a period of many years. Joseph's story demonstrates that these learning styles and study habits can translate to secular studies as well.

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Conferences 2003…

AAMT Australian of Association Mathematics Teachers	Brisbane, Australia	January 13-17
MAA-AMS The Mathematical Association and American Mathematical Society Joint Meeting	Baltimore, MD	January 15-18
AMTE Association of Mathematics Teacher Educators	Atlanta, GA	January 30-Febuary 1
CERME 3 European Society for Research in Mathematics Education	Bellaria, Italy	February 28-March 3
NCTM National Council of Teachers of Mathematics, Research Presession & Annual Conference	San Antonio, TX	April 7-12
Mα The Mathematical Association	Norwich, UK	April 12–15
AERA American Educational Research Association	Chicago, IL	April 21-25

In Focus… In God's Image and Presence: Some Notes Based on an Enactive View of Human Knowing

Thomas Kieren

I was recently at the Edmonton Folk Music festival and listened to Kathy Mattea sing a song which she calls "What if God were one of us?" I think any practicing Christian such as myself or a person of any deliberate spiritual persuasion would find this song interesting. In one small part of it she raises the question of the consequences of observing God in our human world. One consequence of noticing the presence of God in one's world would be that one would necessarily have to face up to the question of believing and all that would entail.

Mattea's song reminded me of two questions of faith: What does it mean to be made or to live in the Image of God? What does it mean to be in God's presence, and in what ways might we see or recognize that presence and live in it? The purpose of this informal essay (these notes) is to look at these two issues (which for me are issues of faith) from the point of view of concepts and consequences of the work I have done for many years in studying human knowing, particularly mathematics knowing. The many people with whom I have worked over the last 10 or 15 years and some of the authors we have studied collect these concepts under the rubric of an "enactive perspective." At its roots, this perspective is scientific and biological. It is certainly not a theological perspective. But to paraphrase an aphorism attributed to Einstein: science without religious or spiritual thought is lame; religious or spiritual thought without scientific thought is blind. Perhaps what I have to say will add perspective to what you already know and believe.

I am not writing this as a piece that is in any way rigorous. I will not be giving the usual citations or references. I will also not try to offer scientific backing for the ideas or the implications I am making here. I certainly will not be writing or acting in a theologically sophisticated way. This essay is based on many years of work studying and inter-acting in my own systematic way with many persons doing mathematics. And what I say is simply a reflection upon the fact that my work is not disconnected from who I am or from who I am as a religious or spiritual person.

Thomas Kieren is a Professor Emeritus at the University of Alberta, Edmonton, Canada. He has been a teacher of mathematics and of mathematics teachers and their students for over 40 years. He has a long standing interest in the constructive mechanisms persons can be observed to use in mathematical knowing acts; in mathematical understanding as a dynamical phenomenon (with Susan Pirie); and in knowing (especially mathematics knowing) and mind as coemergent, fully embodied phenomena.

Even were I to provide you with scientifically or theologically rigorous statements they would be statements I would choose to make from a view which Maturana calls objectivity-in-parentheses. In making them I am not trying to compel you to accept them because they must be necessarily and universally true in the sense that they tell you about the way the world or God really is. Yet they are not subjective. However, they are necessarily incomplete in at least two senses: They are *multiversally* incomplete in that there are other ways of thinking about or explaining the phenomena which are not accounted for and are likely not even anticipated by these views; that is, they are part of a multiversal rather than a universal accounting of things. Secondly, they are *occasionally* incomplete. Even for me as I think or say or write them or for others as they read them, they hold the possibility of recursively occasioning other thoughts which will necessarily "re-write" these.

With these caveats, I turn now to the substance of what I want to say—first to concepts that I am drawing from my enactive work on knowing and then to attempts to see what implications that thinking has for the questions I raise above.

Some aspects of an enactive view of knowing

At the heart of what I have to say is the view that knowing occurs in action in the temporal "now" at once being determined by one's lived history and providing possibilities for future occasions for knowing. In such knowing humans express their nature as auto-poietic in the sense that they necessarily transform "inputs" from the world of their existence for their own use and are closed in that operational sense. In engaging in both developing this point of view and in trying to see its consequences for learning and teaching particularly in the area of mathematics, I feel I have been doing what Varela would call "laying down the path in walking". Of course I have not laid down this path alone or unaided. My guides and companions have been many[1] but that is another story.

More to the point of this essay, such knowing in action is identified with doing. As suggested above, such doing is always determined by the structure and lived history of the person. By the same token, that structure is necessarily plastic and in doing one is always changing that structure. Such changing of one's structure can be observed as learning. Hence knowing, doing, and learning are identified. Since failure to either act or change is concomitant with ceasing to be the organism one is, knowing, doing, learning, and living itself are identified. Perhaps it is this identification that is at the heart of the "elan vital" of humans as well as other living things.

This view suggests that there is no mind/body dualism. Knowing is *fully embodied*. Such embodied knowing has many dynamics which occur all-at-once. There are the internal structural dynamics of the person whereby he enacts a world and is changed by such enaction. There are social/inter-actional dynamics whereby in knowing one is observed to be inter-acting with the world around one and with others in it. This coupling with others and other-ness can be observed to have a conversational dynamic. The nature of the conversation (and one's participation in it) is determined by the structure and lived history of the individual while at the same time that structure is being changed by participating in the conversation. Thus we say that knowing is neither an adequate representation of a pre-given "world," nor is it simply an act of creating a world for oneself. While the knowing/doing is determined by the person, it is also co-determined by the other/otherness with which that knowing occurs. Thus knowing is neither caused by the world nor does it simply emerge from the structure of the person. It is *coemergent*. Another dynamic of knowing is a cultural one. That is, in doing/knowing one is participating in or is embodied in both contemporary and historical practices of the communities in which one exists.

If such knowing is not caused by the world, but coemerges with it, how does the world come to influence our actions, our structures, indeed who we are? For example, in watching children working on a mathematical task using some materials or a computer world an observer might say, "Oh that action pattern by the child was 'occasioned' by the materials." What is meant by this comment? We say the knowing/doing of the person is occasioned by some feature of the other or otherness when the observer sees the action in a way that is relatable to the feature or in the presence in some way of the feature, and when the observer sees the person select or take up that feature and in some way modify it for its own use. Perhaps this occasioning is even clearer when one observes an adult interacting with a young child using language. Suppose one notices that the child uses a word or phrase that the adult has used. It is clear that the child would not use that particular word or phrase had she never experienced its use by others. But at the same time it is clear that the adult could not cause the child to use the word or phrase, and it is easy to see that the child is using the word or phrase in a way that reflects both her own capability/structure and her own intents.

The diagram below is taken from the work of a colleague of mine, Elaine Simmt. She devised the diagram just to show how the knowing/doing of an individual (I) arises in inter-action with or can be observed to be occasioned by others or otherness (O).

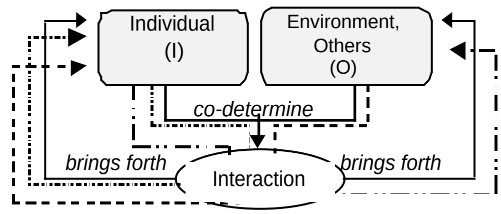


Figure 1. Model for observing knowing in action.

There are many features of this diagram which show the consequences of thinking about knowing in enactive terms. First one sees that the interaction itself is co-determined by the individual and the otherness. Following the path indicated by ------- shows how some feature of O is selected by I and is either consciously (Les Steffe would say through the process of self-regulation) or unconsciously (Steffe would say through the process of auto-regulation) reflected upon and transformed, thus also allowing for the change in I's structures, schemes and possibilities for action. This path illustrates the pathway by which O is observed to occasion knowing in I. Suppose we ignore the right hand side of the Simmt diagram—except if one is thinking of an external re-presentation constructed by I—and focus on the pathway indicated by ············. Here a person, through re-presenting her or his thinking in some way, provides a continuing occasion for her or his own knowing and doing. Thus persons can be observed to occasion their own learning (and I am sure this occasioning also occurs in many forms which are not observable.)

There are two more features of this diagram that are germane to the discussion in this essay. The third path indicated by —··—·· shows another feature of coemergence. When the individual acts, this action necessarily provides the possibility of an occasion for O to change as well. Thus in this model human knowing takes on a necessary ethical dimension in that one's knowing has implications for the environment in which it occurs and not just for the individual her or himself. Finally, if one thinks of this inter-action as occurring and re-occurring, then this inter-action allows for an increase in the cognitive domain or the domain of possible knowing in that both O and I change, allowing both new capabilities in I (changed structure) and new possibilities for future actions to be occasioned.

Maturana and Varela have captured such knowing by saying that it occurs through an individual bringing forth a world of significance with others in a sphere of possibilities for action. Such actions can be of many types. With all living beings we share knowing as physical action—that which is necessary to live and survive. A second level of action is that which occurs with others. Maturana calls this a consensual coordination of actions or a linguistic action. Humans share this capability with many animals. Maturana claims that what is unique to human knowing occurs using the next level of knowing. He calls this languaging, which he sees as a consensual coordination of consensual coordinations of action. Such actions provide the basis for recursively making distinctions in languaging, which is central to human knowing actions. This recursivity not only implies that such acts of knowing use the "results" of previous acts of knowing as "input," but that such distinctions also change one's structure or the way that one was able to know and act previously or the meaning of that previous knowing/doing.

As suggested previously, such languaging occurs in inter-action with others/otherness and through conversations with others that allow the cognitive domain to expand. Such knowing actions are governed by a proscriptive rather than a prescriptive logic. That is, whatever is not wrong is good enough or allowed. Further, such knowing actions are necessarily—in humans at least—intertwined with emotioning, or the inclinations from one's structure or lived history to act in particular ways. Knowing actions affect emotioning as well.

We can summarize the above as follows: knowing is conceived not as a thing, an acquisition, or simply in terms of an external artifact; it is conceived as occurring in action in the temporal now. While that knowing action is determined by the structure of the individual knower, it is coemergent with the space, the otherness, and the knowing of others with which it occurs. Then from the point of view of an observer, this knowing occurs in an inter-action. Through that inter-action the knowing of an individual may be occasioned by their own previous knowing through various re-presentations, by elements in the environment or otherness, by the knowing acts of others or artifacts of them, and also by aspects of the culture (e.g., books) in which they exist and know. Such knowing is construed not as responding to the environment or even as problem solving *per se* (although both responses and problem solving may be observed); it is construed as bringing forth a world of significance. Humans' knowing, while it can and does involve physical action, occurs in languaging and occurs in an environment of languaging and distinctions in language. Knowing may be characterized as occurring in a coupling or a conversation which is necessarily affected by the nature of the structure of and knowing actions of the individual(s) in it; in turn, the conversation affects the knowing actions and hence the plastic structure of the individual.

To the extent that such propositions above explain or explicate knowing, it must be clear that knowing cannot be reduced to such propositions. As Goethe suggested long ago, theory is gray while life is green and vibrant. In applying such concepts to a discussion of our life in the image and presence of God, it is also important to know that such a discussion—even if I were to carefully elaborate all points (which I won't)—would necessarily be incomplete.

Knowing God

The Image of God

It is a tenet of Judeo-Christian faith that we humans are in some way created in the image of God. Since it is another tenet that God is a spirit or of a form other than us, the nature of this image cannot simply be a physical one such as portrayed by a mirror. Since knowing actions are determined by our biological organization and its structural relationships as well as continually changed through our embodied knowing actions, what are the ways in which our being in the image of God affects the possibilities for our knowing?

Mathematics gives us interesting metaphors for image. The first is that we are in a one-to-many relation with God and hence are in God's image space in that way. That is, we are in some specially defined pairing or relationship with God both as species and individually. This suggests one may ask about the inverse relation; that is, the relation that "maps" humans to God. This idea raises questions: Are there many such relations? Is such a relation (what we might observe as knowing God or living occasioned by that knowing) universal, or is it in some way unique to each person? That is, are we part of a large conversation with God, or is that conversation unique to each person, or both, or something else?

A second metaphor for being in the image of God for me comes from thinking about the Mandelbrot set which is fractal in nature. You are likely familiar with pictures of this set which can be generated (but never completely) by iterating certain rather simple mathematical procedures and using particular representational rules to show the set and its boundary regions. For me what is important about this set is that it has a characteristic shape—I see it as a beetle. But when one zooms in and looks at various boundary areas, this fractal set is infinitely complex and varied. The mathematician and artist Peitgen has generated hundreds of beautiful images of the boundary regions of this set which portray it as having infinitely varied and rich characteristics. But across all levels of scale the whole of the Mandelbrot set reappears in all of these rich boundary regions; like all fractals it is self-similar across scale. Using one's imagination, one can think of humans as fractal filaments in the "image of God"—itself a fractal. Like Peitgen's fractal portraits, we each are rich and beautiful in our own way. But perhaps it is we humans who also can observe that God is also "in us" in the way that the beetle-like Mandelbrot set recurs in its own way in any fractal filament at any zoomed-in level of scale. This analogy is another way of thinking about being in God's image: not only are we paired with God in some special relation; we are a part of the infinite eternal "life" of God. Furthermore, if we observe carefully in certain ways, we can sense God in us as well.

Turning from these mathematical musings to one which is more directly related to the previous part of this paper, I consider what is unique about human structure which might illustrate our image nature. It is certainly not the ability to act intelligently or even to take linguistic actions; we share this characteristic with many other animals: their intelligent, usually physical, actions in a space in the temporal now can be observed to be quite sophisticated or at least complex. As suggested by the enactivist view on human knowing discussed above, such knowing occurs in languaging and particularly in the possibility with others to recursively and co-recursively make distinctions in such languaging. This observation raises the question: is it this unique human ability to make such recursive distinctions part of being in God's image? While all such knowing is under-girded by simpler embodied and even physical actions, we sometimes—in fact many times—lose track of or are blind to the physical inter-actions from which come our later distinctions in languaging. These distinctions suggest that we are uniquely equipped to know God. Through our various religious cultures and their artifacts (writings, icons, symbols), we may be occasioned to make such distinctions for ourselves and thus know of the presence of God. But in so doing we may be blind or even unaware of the total embodied relationship to God out of which such distinctions initially arose. Thus this unique ability to abstract provides us with sources both of insight and blindness. And of course simply having the capability or the structure to make such distinctions in no way causes humans to make distinctions that make us aware of God's presence—this phenomenon is shown over and over again in many historical religious narratives.

The presence of God

Supposing that we have the structural characteristics and dynamics to know God, how can we come to live our lives in God's presence? As suggested in the enactivist view of knowing and living, to observe such knowing actions is necessarily to observe and specify the space in which they occur. That is, life does not occur simply through the characteristics of the living or autopoietic thing. The living one (actor, knower) must exist in an energy-rich environment from which the living one selects and transforms elements; in that knowing action the living one is itself changed. With this view, one can observe for oneself that God is the heart of the energy-rich otherness with which and in which we exist. Of course, one need not be conscious of this idea to be alive, know, learn or do. And as part of our process of living one can be aware of the process of selection and transformation of elements from our environment (e.g., air) which do not "need" God for their explanation.

But what if one is conscious of the presence of God as the continuing potential occasioner for good acts of living particularly in languaging (even if we are blind to or have other explanations for the physical aspects of our living)? If we were aware of that occasioning and the inter-action with God through which it occurred, using the ideas portrayed in the Simmt model above, we could expand our domain of possibilities for our lives as well as the spiritual and cognitive domain in which we exist and bring forth with others. That is, the conscious awareness of God as an occasioner of at least certain of our actions changes the way in which we can live—for the better—both by changing us and our structures and by changing the space in which we live and our sense of responsibility of that space and for others in it.

If one takes the view suggested in the last paragraph, one could portray an individual human as being in some way in conversations with God on a number of critical aspects of our lives. Of course both biological and phenomenological analyses shows such conversations to be fragile in that they break down or no longer exist whenever one or the other sides of the conversation chooses to leave it or takes actions to break it. From our own lived histories we all have stories of such broken conversations that often lead to broken relationships which are never recovered. Both parties turn away. Hence regardless of the actions of one or the other, neither the conversation nor the relationship can be taken up again. This phenomenon leads to another feature of the presence of God, which one can observe through this lens of knowing in a particular way. In the histories of religions, in the writings and theologies of them, and in the experience of many of us, there are instances of individuals or groups breaking their conversation with God and their relationships with God through their actions and words. Many times this break in relationship is permanent; in terms of our fractal metaphor, the person no longer experiences herself as part of the eternal life of God. If this broken relationship were with another human or even with something in our physical environment which we had destroyed in our actions, it may well be that no matter what she did, subsequently she could not get back into conversation with that other. If we observe God as the continuing occasioner for our lives, however, we can see that no matter what we did to break the conversation, returning to that relationship and that sense of being is possible. That potential does not mean that our previous actions held no consequences—in fact the model of knowing portrayed above suggests that whether we know it or not our knowing actions have the potential to occasion changes in the other or the otherness as well as changes in our own structures. It simply means that coming into a conversation which allows one to sense the presence of God or returning to that conversation is always possible.

Of course all of these musings beg the question of how we come to or return to an observable knowing relationship with God? In an enactivist view of knowing such action is observed as embodied in the structural dynamics of a person, in the interactive social dynamics with others (in fact bringing forth any world of significance occurs with others), and in the cultural dynamics with the practices of current and historic communities all-at-once. Thus each of the embodiments provides opportunities for relationship. An individual can, through various personal practices (e.g. prayer, meditation, reflection, journaling...), come to be in or enrich one's conversation with God. Second, bringing forth a world entails doing so with others—the presence of God in our lives often becomes most evident and rich in inter-action with or through the acts of others. Finally, engaging in the rituals and practices of a community of faith or engaging with the writings related to that community or other communities of faith are also sources of occasions which allow one to be in conversation with God. None of these are new suggestions. I make them just to show how they can be seen as fitting with the possibilities in human knowing.

Finally, I remind the reader that these views on knowing, their implications for knowing God, and even suggestions for practice (even if I were to elaborate them in detail) are incomplete. In part this incompleteness is both a scientific and a metaphysical necessity. There are other views and other practices that are not contemplated in this discussion. It simply is an invitation for you to think again about knowing, about knowing God, and about one's subsequent actions. This discussion await further thinking and action which it might occasion. Such thoughts and actions will necessarily change these ideas for you, for the community in which you live, and even possibly for me.

To frame this essay, perhaps these notes on the possibility of knowing God's presence raise the other question: What if we were one with God?

[1] This note gives a brief sketch of my guides and companions. (Of course they did not always realize that they were companions in the particular enterprise I am discussing here.) One source of my thinking about knowing has come through working with colleagues whose work is related to or grew out of radical constructivism in one way or another: Ernst von Glasersfeld, Les Steffe, Jere Confrey, Pat Thompson, and Paul Cobb come to mind. I have been deeply influenced by the writings of von Foerster; Maturana; Maturana and Varela; Varela, Thompson, and Rosch; and Northrup Frye. In that reading I have been supported and challenged by colleagues Al Olson, Sandy Dawson, and John Mason. Finally I have worked for many years on these ideas with Susan Pirie, Brent Davis, Dennis Sumara, David Reid, and Elaine Simmt. I'm not sure any of them would agree with the implications I draw here but I am indebted to them nonetheless. And as acknowledged earlier I have done this work at least knowing of and observing being occasioned by the existence of God in my life

Article Review…

A Critical Question: Why *Can't* Mathematics Education and History of Mathematics Coexist?

Kevin Nooney

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science and Education, 10*(4), 391-408.

Fried's abstract of his article (p. 391):

Despite the wide interest in combining mathematics education and the history of mathematics, there are grave and fundamental problems in this effort. The main difficulty is that while one wants to see historical topics in the classroom or an historical approach in teaching, the commitment to teach modern mathematics and modern mathematics techniques necessary in pure and applied sciences forces one either to trivialize history or to distort it. In particular, this commitment forces one to adopt a "Whiggish" approach to the history of mathematics. Two possible resolutions of the difficulty are (1) "radical separation" – putting the history of mathematics on a separate track from the ordinary course of instruction, and (2) "radical accommodation" – turning the study of mathematics into the study of mathematical texts.

Michael Fried makes a confusing case that combining history of mathematics with mathematics education is inherently difficult, if not impossible. Initially Fried's argument seems to rest on the argument that mathematics educators' commitment to modern mathematics makes mathematics education incompatible with the history of mathematics. However, the bulk of Fried's discussion concerns the purposes of the historian: Fried asserts that the concerns of the historian render education incompatible with the history of mathematics. In this review I attempt to expose Fried's unsupported assertions by posing questions that need to be addressed in order to fill out his argument. Specifically, I question his assumption that mathematics educators are unavoidably committed to so-called modern mathematics and his claim that the historian is committed to limit his research to understanding idiosyncrasies in the thinking of historical mathematicians.

Fried states that there is no room for the history of mathematics in mathematics education; he also claims that mathematics educators are so committed to modern mathematics and modern mathematical methods that any attempt to either incorporate historical topics or take an historical approach to mathematics teaching is bound to compromise the "true" history of mathematics. For him, most discussions that support an historical approach to teaching mathematics are guilty of endorsing bad history, or more correctly a "Whiggish" history—an anachronistic reading of history in which modern mathematical concepts are ascribed to ancient thinkers.[1] But, Fried asserts, even if this problem of bogus history could be overcome, the commitment to modern mathematics, mathematics which have been proven best for solving modern problems, leaves no room for examining what true history of mathematics must focus on, namely the idiosyncratic thinking of historical figures. This idiosyncratic thinking not only established mathematics completely different from our own, but often led to many "dead ends" and numerous mistakes by historical mathematicians. The historian of mathematics wants and is duty bound to examine the differences between historical mathematics and modern mathematics. The mathematics educator wants and is duty bound to explain and justify modern methods. The purposes Fried assumes for the historian and the mathematics educator are at such odds as to render any reconciliation impossible, or gravely difficult. Fried takes a such a strong stance against reconciliation throughout his argument that one has to wonder what is left for him to support when he closes by saying that his critical examination of "attempts to introduce the history of mathematics in mathematics education should not be interpreted as *opposing* such attempts" (p. 406, his emphasis).

I find Fried's argument confusing and often contradictory. His discussion is deceptively simple and it is in attempting to analyze his position that confusions arise. For example, Fried proposes two solutions to linking mathematics education to the history of mathematics which are based on current attempts that he assumes early in his discussion to be doomed to failure. The apparent contradictions of his argument lie in the many unsupported claims he makes about the nature of mathematics education and the history of mathematics. In this review, rather than argue against his position, I try to expose these unsupported claims and raise questions that I feel Fried should answer before I, or any one else, consider his position well developed and worth accepting.

Fried takes the failure to implement historical approaches in school mathematics as a signal to consider whether it is possible to combine history of mathematics and mathematics education; in this paper he does not consider the possibility of external sources of failure to implement those approaches. Such sources might include the socio-political contexts of mathematics curricula development, the depth of familiarity of mathematics educators with historical topics, the availability of historical materials for educators, etc. I agree that Fried's concern about the possibility of combining history of mathematics with mathematics education is important. However, he takes the position that there is an inherent difficulty in attempting such a combination. I question this position because it rests on so many unsupported claims.

Fried begins by classifying the recommendations made by advocates of incorporating history into mathematics curricula into two basic strategies. As a result of the educational commitment that Fried claims, both of the two basic strategies for introducing history into school mathematics programs are bound to fail. The first strategy, what he calls the *strategy of addition*, involves "historical anecdotes, short biographies, isolated problems, and… does not alter a curriculum except by enlarging it" (p. 392). This strategy is bound to fail because mathematics programs are already over-crowded and allow little or no room for historical enhancement. The *strategy of accommodation* is more thoroughgoing and demands a restructuring of the mathematics curriculum based on historical development and circumstances. This strategy is bound to fail because it forces either an anachronistic reading of history or a history so deeply edited as to be similarly bastardized.

Fried assumes and asserts his position regarding the commitments of the modern mathematics educator without making a case for accepting that assumption: Mathematics educators have an "unavoidable commitment to the teaching of *modern* mathematics and *modern* mathematical techniques" (p. 392, his emphasis). I believe Fried needs to address at least two major questions: Why should we accept that mathematics educators are committed exclusively to modern mathematics? Why should we accept that such a commitment is unavoidable? Fried might respond that the answers to these two questions are obvious from the purpose of mathematics education; however what Fried suggests as the sole purpose of mathematics education is questionable. I will return to this point shortly.

In his closing comments, Fried quotes and accepts Thomas Tymocsko's claim that "pure mathematics is ultimately humanistic mathematics, one of the humanities, because it is an intellectual discipline with a human perspective and a history that matters" (p. 406).[2] Fried compares mathematics to literature, art, and music; they are all expressions of "that vision and inventiveness so much part [sic] of the human spirit" (p. 406). He then states that the "study of the history of mathematics is an effort to grasp this facet of human creativity" (p. 406). His argument suggests that it is *only* through the history of mathematics that one can attempt to grasp mathematics as a creative endeavor. Would he also suggest that one can grasp art, literature, and music as creative endeavors only through studying their histories? I wonder what Fried conceives of as suitable education in art, literature, or music; would he demand a strong separation of studio arts and history of art in a way parallel to the cleft he sees between mathematics education and history of mathematics? Would Fried consider courses in studio arts to constitute art education and not those in art history? Would he claim that there is no room for art history in art education? While I can imagine that discussions and debates are waged over the roles and relative merit of studio arts and history of art courses, I suspect that most of us, and most arts educators, conceive of studio arts and art history as kinds of art education not as kinds of education. Fried seems to assume that learning mathematics and learning about history of mathematics require two separate kinds of education.

Questions arise, then, about Fried's conceptions of mathematics education and mathematics history. He is much clearer about the purpose of the history of mathematics than he is of the purpose of mathematics education. We have to infer both from his claims about the aims and commitments of the mathematics historian versus those of the mathematics educator. The historian is committed to "understand the thought of the past" (p. 398), to understand and examine the "idiosyncrasy of a mathematician's thought or of the thought of the mathematician's time" (p. 400). The mathematics educator, he suggests, is committed only to preparing future scientists and engineers. Ultimately, in Fried's view, these differing commitments prevent an amicable marriage of history and education in mathematics.

The historian's commitment forces the "serious" historian to always begin with an assumption that the past is different than the present and to focus on this difference. The purpose, then, of the historian of mathematics is to study peculiarities of mathematics—what, for instance, makes a text "*peculiarly* Apollonian or *peculiarly* Greek" (p. 400, his emphasis). This study of peculiarities is what makes the historian particularly interested in the "dead ends" mathematicians come to and the mistakes they make, for these are the kinds of things that reveal the peculiarity of the person's thought; these are the things that reveal the human character of doing mathematics. So, while one might succeed in making mathematics interesting, understandable, and approachable, or in providing insights into concepts, problems and problem-solving without history or with an "unhistorical" history, *humanizing* mathematics with history requires that history be taken quite seriously, not as a mere tool, but as something studied earnestly (pp. 400-401, his emphasis).

There are at least three issues at stake in Fried's claim. One is the foundational assumption Fried takes all historical work to rest upon—the assumption that the past is unlike the present. Another is the focus of study Fried seems to demand that all historians take—the study of idiosyncratic thinking peculiar to a particular historical figure in a particular historical time frame. Also at stake is the status of historical work in education—that proper history is *not* to be relegated to being a mere pedagogical tool.

If all historical work rests upon the assumption that the past is different than the present, and rests only upon that assumption as Fried suggests, of what value can history have? If we assume that the experiences of people in bygone times are completely different than our own, what could we hope to gain by examining their experiences? Couldn't anyone claim that dealing with our own present day experience is difficult enough without shouldering the burden of trying to understand the disconnected and unrelated experiences of someone in another era? And if their experiences, their times, and their way of thinking and understanding is completely disconnected and dissimilar from ours, how can anyone from our time claim to understand them in theirs? Fried draws an analogy between mathematics education and teaching literature by saying that "while one learns something about Elizabethan culture by reading Shakespeare, the main reason one reads Shakespeare's works is that they are great in their own right" (p. 401). If there is nothing to be found in common between the Merchant of Venice and the shopkeeper on Main Street, what is the basis for claiming that Shakespeare's works are *great in their own right*? While jokes about codpieces may be lost on the modern reader or viewer, certainly those same modern readers can understand Hamlet's frustration and sense of being betrayed by those around him. Without some sense of relevance to the modern viewer, *Hamlet* would simply be a collection of odd movements and sayings.

With Fried's insistence that the historian's primary concern is with time dependent peculiarities, what prevents history from becoming little more than a collection of exotic trinkets? (Perhaps though, that is all history is for Fried.) If the only criteria for, or if the fundamental assumption of, the historian is difference, what counts as different? What, and where, is the line that demarcates the past and the present? Do we rely on arbitrary boundaries in terms of years—those who lived within, say, 200 years of the present are assumed to be sufficiently like us to be considered *us*? Or is there some objective measure or process by which the historian can establish *the* difference between *them and us*? It seems that for history to have any relevance, we must assume that the past is, after all, *somewhat* like the present: there must be something from the past that is translatable to the present. While I sympathize with Fried's concern about anachronistic interpretations, I suggest that any historical work that ignores either the similarities or dissimilarities between the past and present and focuses exclusively on one or the other will be severely limited and dramatically incomplete.

Fried claims that any history in which the present is a measure of the past is bad history, or worse, hardly even history at all. My question is how can the present *not* be a measure of the past? Fried rejects the search for the origins of ideas as a major and fundamentally misconceived task for history; in seeking the origins of concepts used in modern mathematics, we will inevitably take away the thoughts of the historical mathematician and "make him think our own" (p. 396); which is to say that we will read into ancient texts modern concepts that were inconceivable at the time. For Fried, it is the Whig history that traces paths (or a single path) from the past to the present, while the truly historical perspective "is the zigzag path of a wanderer who does not know exactly where he is going" (p. 396). But Fried's opposition to reading direction in history seems to blind him to the fact that we cannot *but* examine the past from our own position in the present. The very things that Fried claims are of most interest to the historian of mathematics—"the 'dead ends' mathematicians come to and the mistakes they made" (p. 401)—can *only* be determined as "dead ends" and "mistakes" from the stance of present day mathematical theory and practice. Similarly, what Fried calls modern methods and approaches (that the mathematics educator is unavoidably committed to) and justifies as "the most *powerful* means to solve problems of interest and of importance *in the modern world*" (p. 405, his emphasis) can only be judged "the most powerful" within a historical context. Whatever non-modern methods might be, they proved less fruitful only for the kinds of problems Fried assumes students will eventually face. How can he be sure that they might not be fruitful in the future? Shouldn't students be aware, then, of the mathematics that did not survive in order to enhance their appreciation of the mathematics that educators demand they know?

Fried seems to agree that perhaps they should, but he sees only two possibilities, both of which are extensions of the strategies he previously determined are doomed to failure. One solution is a *radical accommodation*—students would learn mathematics by engaging directly with historical texts (as is done in certain "great books" programs.) The other is a *radical addition or radical separation*—students would have a history of mathematics track parallel to their standard mathematics courses. These "radical" solutions seem to have their basis in Fried's aversion to the "use" of history in mathematics education. Fried is very concerned that history is to be studied in its own right—he seems to deny the same for mathematics.

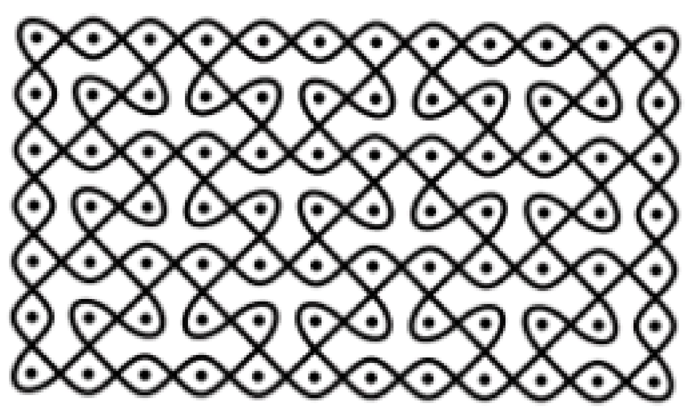
Fried's conception of mathematics education appears to be limited to delivering the useful, powerful mathematics that will prepare competent scientists and engineers. In fact Fried most strongly implies this limited view of mathematics education when he questions whether the humanizing of mathematics through the radical accommodation approach "satisfies the other component of the mathematics teacher's commitment, namely, that students learn to do the mathematics of science and engineering" (p. 401). Does Fried really expect us to accept that the sole purpose of mathematics education is to prepare potential scientific workers in the applications of mathematics? Are we to accept both that mathematics is a human endeavor, with a history that matters, *and* that mathematics is only for the use of scientists and engineers? Would Fried expect a mathematician not to be offended by the implication of his claim that history is not to be used but that mathematics is? Fried claims (probably rightly so) that the commitments of the mathematics educator and of the historian of mathematics make the relationship of each to the history of mathematics quite different (p. 398). Can we not make the same claim about the differences of commitments (and hence the differences in relationship to mathematics) between the mathematician and the mathematics educator? If we take Fried's suggestion that we radically separate mathematics education from the history of mathematics (and hence separate the mathematics educator from the historian), should we demand a similar separation between the mathematics educator and the practicing mathematician?

Ultimately, Fried's argument seems to more about territorial boundaries than about the possibility of infusing mathematics education with historical understanding. Fried examined the mathematics of Apollonius in his doctoral dissertation, and when all is said and done in his case against history in mathematics education, he appears to be a historian trying to preserve the sanctity of his esoteric work from being directed toward any kind of utilitarian purpose. Fried's article abstract and opening remarks lead the reader to expect his case to be that the foundational commitments of the mathematics educator render the history of mathematics incompatible with mathematics education. As I read his argument, struggling to make sense of what appear to be his contradictory leanings, I suspect that the case he actually wants to make is that the commitments of the historian render mathematics education incompatible with the history of mathematics.

[1] Fried adopts the term "Whig history" from Butterfield, H. (1931/1951). *The Whig Interpretation of History.* New York: Charles Scribner's Sons.

[2] Thomas Tymocsko was a philosopher who advocated a quasi-empiricist and fallibilist view of mathematics.

This design, also taken from Paulus Gerdes's *Geometry from Africa: Mathematical and educational explorations* (1999), is a variation on the "chased chicken" *sona* sand drawings among the Chokwe in southern-central Africa. It can be found on page 184 in a chapter devoted to a mathematical examination of *sona* drawings. Trace the chicken's path—what do you notice?



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