Student Teachers’ Argumentation Practices in View of Their Conceptions of Proof

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Abstract

Student teachers’ conceptions of the purpose and need for proof in mathematics were found to align with their support for argumentation in secondary mathematics classes. Student teachers’ conceptions of proof were inferred from interviews, while classroom observations provided most of the data about support for argumentation. A modification of Toulmin’s argument diagrams was used to analyze the argumentation. The student teachers provided and elicited different kinds of warrants in arguments of differing complexity within discourse that, on the surface, could be characterized as IRE. In addition, analysis of argumentation by means of Toulmin’s diagrams is shown to be useful even in classrooms where effective argumentation is not the focus of the teacher.
Student Teachers’ Argumentation Practices in View of Their Conceptions of Proof

While the nature of the relationship between argumentation and proof is difficult to articulate, one can view proof as a specific kind of argumentation, as is described by Pedemonte (2007). Argumentation and proof share a similar dependence on justification, which Yackel and Hanna (2003) describe as giving reasons for a mathematical action or statement “in an attempt to communicate the legitimacy of one’s mathematical activity” (p. 229). In a proof\(^1\), these reasons must be mathematical in nature, building upon one another deductively with an axiom system. In argumentation within a classroom, however, the reasons given for an action or statement may not be mathematical in nature. The collective argumentation that occurs in mathematics classrooms has been shown to influence student learning (Krummheuer, 1995), and the role of the teacher within this argumentation is very important (Yackel, 2002). In this study, I investigated a possible relationship between how a teacher thinks about proof and how he or she supports collective argumentation in a secondary mathematics class.

Much of the research on argumentation in mathematics education has investigated the phenomenon in classes in which the teacher is part of the research team or is working closely with the research team to design lessons and facilitate argumentation among students. These studies were generally conducted in elementary and middle school classrooms with a conscious emphasis on argumentation on the part of the teacher, and the collective argumentation that is described is characterized by rich student-to-student (and student-and-teacher) interactions (e.g., Forman & Ansell, 2002; Krummheuer, 1995; Zack & Graves, 2001). One goal of this study is to provide a picture of what argumentation looks like in regular public high school classrooms where its facilitation is not an explicit goal of instruction.

\(^1\) I consider a proof to be a logically correct deductive argument built up from given conditions, definitions, and theorems within an axiom system.
A definition of proof as a logically correct deductive argument highlights the connection between proof and argumentation as one thinks about a proof as a particular kind of argument. However, considering the relationship implied by this definition does not clarify whether engaging in collective argumentation might lead to improved ability to construct proofs or how a teacher’s knowledge of proof may influence his or her practice with respect to argumentation. Given the dependence of argumentation and proof on justification (as described by Yackel & Hanna, 2003), it is reasonable to believe that a teacher’s conceptions of proof may relate to how he or she facilitates argumentation in his or her classroom. A second goal of this study is to explore a possible connection between one aspect of a student teacher’s conception of proof and his or her support for collective argumentation.

This study extends the current literature on argumentation and attempts to connect the work on proof with research on collective argumentation by examining the following questions.

- How do prospective secondary mathematics teachers support collective argumentation in secondary mathematics classrooms?
- What characterizes the relationship between the argumentation observed in a particular classroom and the prospective teacher’s conception of proof and justification?

**Conceptual underpinnings**

Krummheuer (1995) describes collective argumentation as “a social phenomenon, when cooperating individuals tried to adjust their intentions and interpretations by verbally presenting the rationale of their actions” (p. 229). Collective argumentation, as described by Krummheuer, embodies much of the spirit of both the Reasoning and Proof Standard and the Communication Standard in the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000). These two process standards form part of the basis for current
recommendations about how students should interact with mathematics, and thus how teachers should conduct their classes. Within argumentation as conceptualized by Toulmin (1958/2003), an individual provides claims, data, warrants, backings, qualifiers, and rebuttals in an attempt to convince an audience of the validity of a particular claim. The Reasoning and Proof standard emphasizes that the ways in which mathematical validity is established within mathematics classrooms are important. One way that mathematical validity may be established in a classroom is through collective argumentation. Collective argumentation occurs when participants in a discussion act in such a way as to work together in the construction of an argument. This working together requires communication about mathematical ideas in the spirit of the Standards, and will hopefully result in consensus about the mathematical validity of those ideas. The role of the teacher is quite important within these discussions.

Teachers may influence students’ learning through their actions with regard to classroom discourse (Hiebert & Wearne, 1993; Yackel & Cobb, 1996), in particular, through initiating the negotiation of norms with regard to collective argumentation. A normative standard of argumentation is a norm observed within one part of the mathematical activity in a classroom: the collective argumentation surrounding mathematical ideas. According to Cobb, a norm “refers to patterns in collective activity within a classroom” (2002, p. 190). Norms are not established but are “interactively constituted by the teacher and students” (Cobb, 2002, p. 190), making the teacher’s role crucial in the negotiation of norms (McClain, 2002). Of particular interest when considering normative standards of argumentation are sociomathematical norms. Sociomathematical norms include, among other things, what has been established in a classroom as an acceptable mathematical explanation or justification (McClain, 2002; Yackel & Cobb, 1996).
When negotiating sociomathematical norms, including the particular normative standards of argumentation that address acceptable mathematical explanations or justifications, the teacher may act as a representative of the external mathematical community within the classroom mathematical community (Yackel & Cobb, 1996). In an ideal setting, when the teacher takes on the role of a representative of the mathematical community in negotiating normative standards of argumentation in a mathematics classroom, he or she would model and support ways of arguing—including providing relevant and mathematically appropriate data, warrants, and backings for claims—that are consistent with how knowledge is constructed in the larger mathematical community². Because knowledge construction in the field of mathematics is dependent on proof, a teacher who is acting as a representative of the mathematical community would attempt to facilitate in an accessible way argumentation that is consistent with acceptable mathematical justification that could eventually lead to proof. The teacher who acts as a representative of the larger mathematical community would not necessarily require every argument be a deductive proof. Instead, he or she would support and model data, warrants, and backings for claims that are acceptable in the field of mathematics (Wood, 1999).

It is probably not the case that every teacher consciously acts as a representative of a larger mathematical community in his or her classroom practice. Some teachers may not think of themselves in this way, and others, while they may consider themselves to be part of the larger mathematical community, may be influenced by a host of other factors, such as organizational and assessment issues and conceptions of teaching and learning (see, e.g., Ball, Lubienski, & Mewborn, 2001). In particular, the teacher’s actions as a representative of the mathematical community.

² Exemplary classrooms, in which the teacher assumes or intends to assume this role, are illustrated in much of the reported research on classroom argumentation (e.g., Forman & Ansell, 2002; McCrone, 2005; Yackel, 2001; Zack & Graves, 2001).
community with respect to argumentation are mediated by the teacher’s understanding of justification and proof, as this is part of his or her conception of proof, which is part of his or her conception of mathematics. That is, the teacher represents only a mathematical community that is consistent with his or her conceptions of and experience with mathematics, because he or she is not able to represent a community of which he or she is not aware or with which he or she has no experience. Thus, the data, warrants, and backings for claims that he or she supports or models within the classroom may tend to align with the teacher’s conception of proof and justification, moderated by other personal, classroom, and institutional factors.

In this study, I critically examine two student teachers’ support for argumentation in light of the structure conceptualized by Toulmin (1958/2003) and adapted by Krummheuer (1995) and compare this to the student teachers’ conceptions of proof in order to open a dialogue about the relationship between teachers’ conceptions of proof and their classroom practice. In particular, I argue that the warrants provided and prompted by the student teachers in this study align with these student teachers’ conceptions of the purpose and need for proof in mathematics. Building from this result, I suggest that teachers’ beliefs about the purpose and need for proof in mathematics must be considered in the mathematical preparation of teachers in light of the crucial importance of argumentation in classroom discourse practices.

Methods

The findings reported here are from a multicase study involving three student teachers who were enrolled in their culminating field experience at a large university in the eastern United States. All three student teachers majored in mathematics with a teaching option and earned grade point averages higher than 3.5 on a 4-point scale. These three student teachers were selected based on their placement in a school and with mentor teachers who, based on the
previous experience of the student teaching supervisor, would allow the student teachers to
design and implement instruction with a maximum of independence from methods preferred by
the mentor teachers.

In order to examine the student teachers’ support for argumentation, I observed each
student teacher as he or she taught classes in algebra, geometry, or calculus. These observations
occurred in two to three day blocks approximately every other week for seven weeks, beginning
several weeks after the student teaching experience had begun. Each student teacher was
observed approximately eight to ten times. During the observations, I audio recorded the class,
later transcribing what the student teacher said, and took detailed field notes, recording student
comments and anything written on the board or overhead. These transcriptions and field notes
were later combined and used as the record of what occurred during class for the purposes of
analyzing the student teachers’ support for argumentation.

To elicit the student teachers’ conceptions of proof, I interviewed each student teacher
twice, using semi-structured interview protocols. These interviews were audio and video
recorded and then transcribed. The first interviews were completed prior to any observations, and
the second interviews occurred after the observations were concluded. During the first interview,
each student teacher was asked questions about his or her previous experiences with proof (e.g.,
Describe a recent proving experience in which you felt successful.), asked to give his or her
opinions about proof and mathematics (e.g., How important do you think proving is in
mathematics?), and asked to complete several tasks involving constructing and critiquing
arguments or proofs (e.g., he or she was given an argument that the square root of a positive
integer is either an integer or it is irrational and asked if the argument proved the statement). At
various points in the interview, the student teachers were asked to answer questions from the
perspective of a student of mathematics and from the perspective of a teacher of mathematics.

However, not surprisingly, they tended to switch between these roles rather swiftly, and it was not always possible when analyzing their responses to clearly distinguish between the two. The interview schedule for the first interview was the same for all participants.

The second interviews were designed to elicit the student teachers’ beliefs about proving in the context of teaching mathematics and to determine each student teacher’s belief about the influence of his or her mentor on his or her practice. The interview schedules were constructed to be similar for all participants but based in their classroom practice. For instance, each student teacher was asked to describe an ideal answer to a question he or she had posed to students on a quiz or test that involved asking why or an instruction to justify an answer. Other questions, such as what counts as evidence of a strong justification, were asked of all participants.

I interviewed each mentor to examine the extent of his or her influence on the teaching practice of the student teacher. These short interviews were audio recorded. During these interviews, I asked questions about interactions between the mentor and student teacher (e.g., What advice do you give about engaging students in talking about mathematics?), and I asked questions about each mentor’s normal classroom practice (e.g., To what extent do you ask students to justify their answers?).

In addition to the video and audio recordings and written artifacts from the interviews and field notes and audio recordings from the observations, I also collected relevant lesson plans, worksheets, tests, and quizzes with answer keys from each student teacher. The data corpus, then, included interview data from each of the student teachers and his or her mentor, observation data, and lesson and unit artifacts. Details of data analysis appear in the next section.
This paper reports on one aspect of the larger study, the relationship of a student teacher’s view of the purpose and role of proof in mathematics to his or her facilitation of argumentation in a particular classroom. To this end, I focus on two student teachers, Karis and Jared, and rely primarily on data from their interviews and classroom observations. Karis, who taught calculus, and Jared, who taught algebra, provide examples of student teachers with very different conceptions of proof, particularly in terms of their views of the purpose and need for proof in mathematics. Karis viewed proof as important for explanation and insight, while Jared viewed proof as important for knowing how to do things. These different conceptions of proof were apparent in the different ways they supported argumentation in their classes. Further details about the third teacher may be found elsewhere (Conner, 2007, 2008).

Analytic Framework

The analytical framework underlying the analysis of the data has two key components. One component deals with the structure of the collective argumentation observed in the mathematics classrooms and how the student teachers supported that argumentation. Krummheuer’s (1995) adaptation of Toulmin’s (1958/2003) framework provided the basis for this analysis. The other component addresses the student teachers’ conceptions of proof and is based on a synthesis of the literature on proof and proving.

Argumentation

Krummheuer’s (1995) adaptation of Toulmin’s (1958/2003) description of argumentation provided a starting point for the analysis of the collective argumentation that occurred in each student teacher’s class. Toulmin described an argument as consisting of a claim, data, warrants, backing, qualifiers, and rebuttals, and posited that the structure of an argument can be diagramed.

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3 All names are pseudonyms.
as illustrated by Figure 1. Each element of an argument (except the claim) may be either explicit or implicit, and some arguments may not have qualifiers or rebuttals. Krummheuer suggested that the elements of an argument may be contributed by various members of a classroom community rather than by one person who is attempting to convince an audience (as was intended in Toulmin’s description). Using “the core” (p. 251) of Toulmin’s model, Krummheuer diagrammed the claims, data, and warrants of arguments that occurred in elementary classrooms to examine the learning of mathematics in the context of collective argumentation. In the following paragraphs, I define each element of an argument and illustrate the elements and relationships between elements with an example from Karis’ class.

**Figure 1.** Toulmin’s diagram of an argument (adapted from Toulmin, 1958/2003, p. 97).

In general, an argument can be identified by the presence of a claim; one must have something for which to argue. According to Toulmin (1958/2003), a claim is a “conclusion whose merits we are seeking to establish” (p. 90). An example of this from the current study comes from Karis’ calculus class on February 28. Karis and her students conclude that the Mean Value Theorem can be applied to \( f(x) = x(x^2 - x - 2) \) on the interval \( I = [-1,1] \). The statement of the claim starts at the beginning of the episode and is completed after the data and warrant for the claim are given.
Toulmin (1958/2003) describes data as “the facts we appeal to as a foundation for the claim” (p. 90). In the course of the discussion of the above claim, Karis and her students provide the following data: \( f(x) = x(x^2 - x - 2) \) is continuous and differentiable. A question about the validity of a datum elicits a sub-argument. For this example argument, no one in the classroom openly questioned the validity of these data. If someone questioned the validity of the data, or, in a classroom setting, questioned the data in any way, the class would engage in one or more sub-arguments, in which the questioned data would become the claims, and further data and warrants would be provided in support of those claims. For instance, Karis or a student could have asked why \( f(x) = x(x^2 - x - 2) \) is continuous, and the class could have engaged in a sub-argument in which “\( f(x) = x(x^2 - x - 2) \) is continuous” would have been the claim, a statement that \( f(x) = x(x^2 - x - 2) \) is a polynomial function might have been the data, and a statement about the continuity of polynomial functions might have been the warrant.

A question about the relevance of the data to the claim requires the statement of a warrant. According to Toulmin (1958/2003), warrants are “general, hypothetical statements, which can act as bridges, and authorise the sort of step to which our particular argument commits us” (p. 91). Toulmin suggests that data are usually stated explicitly, while warrants are left implicit unless specifically requested. A warrant is requested when the data is challenged in a particular way that demands an accounting of the link between the data and claim. In our example, Karis and her students contributed to the warrant, “The two requirements of the Mean Value Theorem are [the function is] continuous on the closed interval and we can take the derivative on the entire open interval.” (Karis Calculus Class, February 28, lines 8-11, repetitions deleted for clarity). This warrant was given at Karis’ request, not after a challenge to the data, but as a way to ensure that the data were related to the claim.
Like data, warrants may be challenged in two different ways. The validity of the warrant might be challenged, in which case a sub-argument would need to be made to support its validity. The relevance of the warrant in the field in which the argument was being made could also be challenged. For instance, if a student in a mathematics class claimed that the data related to the claim because “that’s the way I learned to do it,” the warrant could be challenged on the grounds that appropriate warrants in mathematics classrooms should have a mathematical basis. This type of challenge to a warrant is really an appeal to the backing of the argument.

The backing of an argument, according to Toulmin (1958/2003), is usually unspecified. Backings are “other assurances without which the warrants themselves would possess neither authority nor currency” (p. 96). According to Toulmin, the kinds of warrants and backings that are acceptable for a particular argumentation are dependent upon the field (such as law, mathematics, sports, or art) in which the argumentation is occurring.

Toulmin (1958/2003) also included two other components of argumentation in his model: qualifier and rebuttal. A qualifier indicates “the degree of force which our data confer on our claim in virtue of our warrant,” (p. 93). An example of a qualifier is “probably” or “presumably” (p. 93). A rebuttal indicates “circumstances in which the general authority of the warrant would have to be set aside” (p. 94). According to Toulmin, a rebuttal is necessary “in all cases where the application of a law may be subject to exceptions, or where a warrant can be supported by pointing to a general correlation only, and not to an absolutely invariable one” (p. 95). The student teachers and their students in this study acted as if their statements and arguments applied without qualification. Thus, while an awareness of these two components of Toulmin’s model was maintained, qualifiers and rebuttals were not found to be useful in characterizing argumentation in these classrooms. This is consistent with Krummheuer’s (1995) adaptation of
Toulmin’s model to mathematics education, as well as how this model is commonly used within mathematics education (see, e.g., Cobb, Stephan, McClain, & Gravemeijer, 2001; Forman, Larreamendy-Joerns, Stein, & Brown, 1998; Whitenack & Knipping, 2002; Yackel, 2002). It is reasonable that within collective argumentation in mathematics classes, these two components could be found to be useful (see Inglis, Mejia-Ramos, & Simpson, 2007; Lakatos, 1976).

I further adapted Krummheuer’s (1995) methods to account for which member of the community was contributing each element of an argument in order to ascertain how the teacher facilitated or supported the argumentation that occurred. As can be seen in Figure 2, I distinguished between contributions by a student teacher, the students, and by the student teacher and students working together. In addition, in order to examine the complex nature of the argumentation that occurred within these high school mathematics classes, I examined both individual arguments and episodes of argumentation. An episode of argumentation consists of the main argument in support of a claim along with any sub-arguments for the validity of data or warrants in the main or other sub-arguments. Figure 2 shows an episode of argumentation with only a main argument. Some episodes of argumentation contain only one main argument, but many contain multiple sub-arguments. In examining arguments and sub-arguments, like Krummheuer, I focused on the core of arguments (claim, data, warrant), since the backings of arguments are usually implicit and dependent on the individual’s framing.

Diagramming the observed argumentation in these ways allowed me to examine the structure and patterns in argumentation in each of the student teachers’ classrooms in order to address each of my research questions. Noting who contributed which elements of an argument allowed me to document how the student teachers supported argumentation. Examining patterns

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4 For more information about framings in the context of argumentation, see Krummheuer (1995).
in the diagrams gave insight into part of my second question, allowing me to compare these patterns with each participant’s conception of proof.

Figure 2. Diagram of argument from Karis’ Calculus class, February 28; ⬣ denotes a student contribution, ⬤ denotes a teacher contribution, and ⬤ □ denotes that both contributed.

Conceptions of Proof

Each student teacher’s conception of proof was inferred from his or her responses to interview questions. Supporting evidence for these inferred conceptions was also provided by examining the lesson artifacts, such as worksheets, tests, and answer keys supplied by the teacher. A student teacher’s conception of proof was conceptualized as a combination of several different elements, including his or her ability to construct and analyze proofs, his or her beliefs about the purpose of proof in mathematics, and his or her affective reaction to proofs and proving. These elements of a conception of proof were derived from the literature on proof and proving.

Much of the research on proof and proving has focused on learners’ abilities to prove and analyze proofs. These studies have focused on what makes proving difficult for students (e.g.,

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Weber, 2001), what convinces students of the truth of a mathematical statement (e.g., Harel & Sowder, 1998), and how students validate texts as proofs (e.g., Selden & Selden, 2003). It has long been established that proving is difficult for students (see, e.g., Almeida, 1995; Bell, 1976; Moore, 1994; Recio & Godino, 2001). A variety of reasons for this difficulty have been investigated, including: their understanding of relevant mathematical concepts (Almeida, 1995; Chazan, 1993; Moore, 1994; Weber, 2001), a lack of knowledge of the principles of logic (Hazzan & Leron, 1996; Knuth, 2002b; Selden & Selden, 2003; Stylianides, Stylianides, & Philippou, 2004), an understanding of proof based on different institutional experiences with proof (Recio & Godino, 2001), and having a view of proof that differs from an ideal or practicing mathematician’s perspective (Almeida, 1995; Chazan, 1993). The perspectives of Moore and Weber, who worked with undergraduate students in attempts to construct proofs, are particularly helpful in explaining differences in a student teacher’s work with proofs in different contexts. Both Moore and Weber concluded that one reason students were unsuccessful in generating correct proofs was that they did not possess the mathematical knowledge necessary for the proof or they did not understand how to use the mathematical knowledge that they did have in a particular mathematical area.

Several authors suggest that proof may have different purposes or functions within mathematics and mathematics education. Hanna (1995, 2000) suggests that, in the classroom, the role of proof is primarily to explain and to promote understanding. Hersh (1993) claims that the primary role of proof in mathematics is convincing, while the primary role of proof in education is explaining. Thurston (1995) suggests that one of the roles of proof in mathematics is ensuring certainty. Various empirical studies have found that learner’s views of proof include these different roles. In a study of preservice teachers’ backgrounds, beliefs, and attitudes about proof,
Mingus and Grassl (1999) found that many of their participants believed the role of proof was to explain how and why concepts work, quoting one participant as saying, “It is easier to remember and understand how to do math if you know why it works and explore why it works for yourself instead of having someone just tell you how to do it” (p. 441).

Several authors theorize that students do not perceive a need for proof in mathematics. Dreyfus and Hadas (1996) discuss students’ lack of a perceived need for proof in the introduction of their study. Hanna (1995) suggests that one role of a mathematics teacher should be to emphasize the need for proof to his or her students. Interviews with participants in two studies confirm that some teachers and students do not believe proof is integral to mathematics: in Knuth’s (2002a, 2002b) study, the secondary mathematics teachers viewed proof as a topic in the curriculum rather than “an essential tool for studying and communicating mathematics” (Knuth, 2002b, p. 84); and some of Almeida’s (2000) participants, in his study of proof perceptions, viewed proof as important only for passing examinations.

Based on a synthesis of these research results, I asked participants interview questions that were set in different mathematical contexts (including geometry and number theory) so that my understanding of their ability to construct and analyze proofs was not limited by the mathematical context. In the analysis of conceptions of proof, I noted whether difficulties seemed to be related to knowledge of the mathematical concepts, knowledge of logic, or a different view or definition of proof. I asked participants why we prove things in mathematics, and paid attention to their statements about the various roles of proof in the analysis of their interviews. I also asked them to respond to several questions first as a learner of college mathematics and then as a teacher of high school mathematics to determine if they held different beliefs about the purpose of proof in college mathematics versus high school mathematics. In
addition to asking why we prove in mathematics, I asked participants how important proof is in mathematics, and specifically how important proving is in high school mathematics. As I analyzed their responses, I paid attention to whether they differentiated between the importance of proof in mathematics and school mathematics, and I compared their statements to their classroom practice.

Within the analysis of data in this study, I analyzed the argumentation and the conceptions of proof separately before beginning to compare the two aspects of this study. I used the diagrammed structure of arguments to describe the argumentation in each participant’s classroom in answer to the first research question. I then examined each participant’s conception of proof by summarizing themes that arose from several passes through the interview data and searching for confirming and disconfirming evidence in the data. Finally, I compared and contrasted each participant’s conception of proof with his or her support for argumentation. At each stage, another mathematics educator examined subsets of the data and offered alternative explanations or interpretations, which were discussed and differences resolved. Principles arising from these conversations guided subsequent analysis of the data.

Results

Karis and Jared provide examples of student teachers with very different conceptions of proof, particularly in terms of their views of the purpose and need for proof in mathematics. Karis viewed proof as important for explanation and insight, while Jared viewed proof as important for knowing how to do things. These different conceptions of proof were apparent in the different ways they supported argumentation in their classes. In this section, I describe each case separately, addressing the research questions for each case within the description. I then
focus on the relationship between their conceptions of proof and support for argumentation in a
cross-case synthesis at the end of this section.

*Karis: A Good Proof is a Good Explanation*

Karis strongly believed that the purpose of proof was explanation to increase the understanding of the intended audience. When talking about justification in the interviews, Karis often interchanged the words “explanation” and “justification.” In a way that was similar to Knuth’s (2002b) description of explanation as providing insight, Karis emphasized that proof should be used in school mathematics so that students would understand the mathematical ideas. Karis consistently demonstrated attention to the audience for a proof, constructing and evaluating arguments only after ascertaining the intended audience for the proof. This focus on proof as explanation for the purpose of understanding was apparent not only in her interviews about proof and proving, but also in her support for argumentation. In the following sections, I will illustrate this aspect of Karis’ conception of proof and describe how her support for argumentation shares similar characteristics.

*Proof and explanation.* Karis both talked about the purpose of proof and justification as explanation and identified explanation as an important part of proving. When asked, “Could you estimate how frequently you ask students to justify their answers in class?” (Karis Interview 2, lines 208-209), Karis replied, “[Do] you mean, when I ask a question, saying things like why, or like explain?” (Karis Interview 2, lines 210-211). When she was asked, “Can you give me an example...of a time when you asked a student to justify something and he or she just gave a really good justification?” (Karis Interview 2, lines 274-277), Karis gave the following reply.

Well, I guess today in calculus they’re doing some more area under a curve stuff, and I gave them an example, uh, just the general parabola, like it was $x$ squared, but from negative two to positive two, so they had two sides, it was the first time they saw one like that, so they had to think of, whoa, what do I have to do differently kind of thing, because
we’re doing inscribed rectangles, so if they used the left endpoints the whole way, then on one side they’re circumscribed instead of, so I asked one of the students to explain, you know, like what can we do? And they gave pretty good explanations of that, explaining like, oh, well, why don’t we just split it in half? And because if I use right endpoints on this side I’m fine, and if I use left on this side it’s fine, so they went through that whole thing, and then another student added on and said, well, look, it’s symmetrical, can’t we just do two times one, so I guess that was a good explanation, they just really saw and like were building off of each other’s suggestions, so I thought that was a good explanation. (Karis Interview 2, lines 278-293, emphasis added)

When asked, “What do you count as evidence of a really good, strong justification?” (Karis Interview 2, lines 297-298), Karis replied:

I guess putting thought into it and using, maybe, like the theorems or concepts or whatever it is we talked about in class, as backing up, instead of just well, this is what I did and that’s it kind of thing, just really showing, well, I remember we talked about this theorem, and this is, I don’t know, just going through and applying the stuff we’re learning and really going step by step, and things like that. (Karis Interview 2, lines 299-304)

This reply was similar to her description of what was involved in proving a mathematical statement: “I guess showing the steps to get to it, I guess starting with what they give you and using whether it’s algebra or reasoning to show the steps between what they give you and what you’re looking for and explaining each step” (Karis Interview 1, lines 134-138). Karis was consistent in her descriptions of proving and justification in that each involved some aspect of explaining and “going step by step” (Karis interview 2, line 304).

Both a focus on explanation as the purpose of proof and a tendency to equate justification with explanation are consistent with previous findings about the role of proof in mathematics. Karis’ use of “explanation” and “justification” synonymously illustrates a difficulty described by Dreyfus (1999). Dreyfus suggested that sometimes an explanation is a proof, and that teachers and students may differently interpret an instruction to explain. Karis’ consistent answers in which she replaces the word “justify” with the word “explain” or describes a justification in terms of explaining suggest that when asking her students to either justify or explain, she intends
for them to provide insight into why something is true. To Karis, this includes both describing a method of finding an answer and appealing to appropriate theorems and definitions as rationales within an explanation or justification. Even though the secondary teachers in Knuth’s (2002b) study did not explicitly mention explanation as a role for proof in mathematics, some of their actions and statements suggested an explanatory role for proof as a way to show how something came to be true. Knuth points out that this view of proof as explanation is different from explanation as a way of “promoting insight of the underlying relationships” (p. 80). Karis’ view of proof as explanation seems to align more closely with providing insight into the underlying relationships or, as Hersh (1993) described, seeing the primary role of proof in a classroom as to “provide insight into why the theorem is true” (p. 396). Hanna (1990) also argued that one very important function of proof in mathematics is explanation, and further suggested that proofs in mathematics classes should be explanatory whenever possible. While Karis did not engage her students in constructing proofs in her calculus class, the argumentation observed in her classroom as well as her description of and work with arguments and proofs in her interviews suggest that she would agree with Hanna’s recommendation.

Students as audience. When directly asked about proof in secondary mathematics classes, Karis contrasted the way proof was used in her learning of school mathematics and the way she intended to use proof in her teaching of mathematics. She said, “So it [proof in my school mathematics experiences] wasn’t used in the same way that I hope to use it, which is more of explanation and helping to understand” (Karis Interview 1, lines 82-83). Karis’ concern for using proof as a way to enhance her students’ understanding is related to a consistent theme in her interactions with proofs in her interviews. When Karis was asked to construct a proof, one of her first questions concerned her audience. She asked, “Am I supposed to aim for a certain age level,
or anything?” (Karis Interview 1, lines 181-182). When told to prove it for one of her peers, she proved the mathematical statement (the sum of the first \( n \) natural numbers is \( \frac{n(n + 1)}{2} \)) by induction. However, when asked what she would have done if the answer had been to prove it to a high school student, she constructed a less formal argument. As she wrote what is in Figure 3, she made the following statement.

So if we just took like \( n \) numbers, and if I line them up backwards, then \( n \) plus \( n \) minus one plus \( n \) minus two back down to one, and then I sum this way, what happens is that I can notice a pattern, and here I’ll get \( n \) plus one, and for these two I get \( n \) plus one also, because it’s \( n \) minus one plus two, which is \( n \) plus one, and here I have three plus \( n \) minus two, so this is also \( n \) plus one, and so I notice this pattern all the way down, so that they’re always \( n \) plus one. And so this would happen then \( n \) times, cause I have \( n \) different numbers, so we could think of it as \( n \) times \( n \) plus one, is what we have. But what we’re actually adding there then is two \( n \) numbers, so you divide by the two. And so that’s why the first sum, the sum of the first \( n \) numbers is \( n \) times \( n \) plus one over two.

(Karis Interview 1, lines 218-230)

Karis used ideas that would be more familiar to a student, such as writing down the sums, adding them together, and then counting in order to find an expression for the sum. Her first argument used induction, which would probably not be familiar to many high school students, while this argument did not rely on such an advanced understanding of mathematics.
Figure 3. The written work from Karis’ argument to convince a high school student that the sum of the first $n$ natural numbers is $\frac{n(n+1)}{2}$.

As evidenced in this and other instances in her interviews, Karis seems to believe that what a proof looks like and what it should contain depends on its audience. She considered the context of an argument to judge its effectiveness. This is consistent with Hanna’s (1990) contention that an acceptable proof is one that is or has been accepted by mathematicians, and that this acceptance is a social process. Karis indicates that her peers (and later, when asked, a mathematician) would expect and accept the argument by induction, while a high school student would tend to accept her argument by rearranging and counting. This view of proof is also consistent with Hersh’s (1993) definition of proof as “a convincing argument, as judged by qualified judges” (p. 389). Karis considered her audience when constructing and evaluating arguments in her interviews. A similar concern for audience was observed in her support for argumentation in her classroom.

Supporting argumentation. While I did not observe Karis proving any mathematical statements in her calculus class, Karis’ support for argumentation did align with her emphasis on explanation for the purpose of student understanding. A typical episode of argumentation in
Karis’ class involved Karis and several of her students. Within an episode, there were usually two to four sub-arguments, although some contained up to eight. A typical argument involved Karis asking a question that suggested a claim should be made, a student or students providing data, and Karis providing the warrant. The claim was often made at the end of the argument, particularly in main arguments involving calculations, and Karis’ initial question that prompted the claim often took the form of writing a problem or question on the board. For instance, in the episode of argumentation described below, Karis had written \( \int \cos(3x + 2) \, dx \) on the board. This was a problem from the homework assignment that was due in class that day, and a student had asked if they could talk about how to do it.

K: Now in this case it’s a little different, because we don’t have something raised to a power, but this is when Mr. A’s way\(^5\) helps you. If you were asked not to find the antiderivative, but to find the derivative, what rule would you have to use?

S: Chain rule

K: Yes. You’d have to use the chain rule because you need to take the derivative of the cosine, but you can’t forget the derivative of the inside. So, if you think about it like that, we know, okay, we would need the chain rule, so that means we need substitution. The next question, then, is what should we substitute for?

S: 3x+2

K: Yes. In the case when you have trig functions, it should always be the inside that you sub for. So it’s still the inside, it’s just that it’s not raised to a power, we’re taking the cosine of the inside, so then, we have to find something for dx, so \( du \), is three \( dx \). But I only have one \( dx \), so I need to multiply by the one-third again, so that one-third \( du \)

\[
\frac{1}{3} du = dx
\]

K: So I go back and sub in. I have the integral of the cosine, instead of putting this in, I use my substitution, so I say the cosine of \( u \) \( dx \) is one-third \( du \), and then, I’m pretty much ready that I can find the antiderivative. So using your chart, what’s the antiderivative of cosine?

---

\(^5\) Earlier in the class, Karis had introduced the idea that if you would have to use the chain rule to find the derivative of an expression, you should use substitution to find the antiderivative of that function, attributing this idea to her mentor, Mr. A.
S: Sine
K: Positive sine or negative sine?
S: Positive.
K: So it’s one-third sine of \( u \), have to put it in terms of \( x \)’s, so it’s one-third times the

\[
\int \cos(u) \left( \frac{1}{3} \right) du
\]

sine of three \( x \) plus two plus our constant. [The board now reads]

\[
\frac{1}{3} \int \cos(u) du = \frac{1}{3} \sin(u) = \frac{1}{3} \sin(3x + 2) + C
\]

K: So the original problem was that we don’t have a rule for what the cosine of three \( x \) plus two is, we only know what the cosine of \( x \) is. So that’s why we need the substitution. Once we do the substitution, then we have the cosine of just the variable, turn back to your chart, see what the antiderivative is, and then your last step is just going back to \( x \)’s, putting your constant in.

(Karis Calculus Class, March 13, lines 82-109)

This episode of argumentation contains one main argument and two sub-arguments, as outlined below and diagrammed in Figure 4. The main argument has the following components (notice the contributor for each of the warrants):

- Claim by Karis: \( \int \cos(3x + 2)dx = \frac{1}{3} \sin(3x + 2) + C \).
- Data by students and Karis: We need to use substitution; substitute \( u = 3x + 2 \); antiderivative of cosine is sine; then go back to \( x \)’s; put constant in.
- Warrant by Karis: Substitution property; chart for antiderivatives.

The first sub-argument has the following components:

- Claim by Karis: We have to use substitution to find this antiderivative.
- Data by students: We would have to use the chain rule if we were finding the derivative instead of the antiderivative.
• Warrant by Karis: Mentor teacher’s rule is if you’d need to use the chain rule to find the derivative, you have to use substitution to find the antiderivative. Later, she also says, “we don’t have a rule for \( \cos(3x + 2) \), just for \( \cos x \)” (Karis Calculus Class, March 13, lines 106-107).

The second sub-argument has the following components:

• Claim by students: We should substitute for \( 3x + 2 \).

• Data: unspecified (This is a trig function.)

• Warrant by Karis: “When you have trig functions, it should always be the inside that you sub for” (Karis Calculus Class, March 13, line 92).
Importance of warrants. Karis’ support for argumentation was characterized by her provision of warrants for arguments. Notice that in the episode above, Karis contributed every warrant. While Karis did not provide every warrant in every argument, there were only three observed arguments (out of a total of 75 arguments) for which students provided warrants without input from Karis. Since a warrant is the part of an argument that connects the data with the claim, providing an explanation of or justification for that connection, this low number of warrants from students seems to be unusual given Karis’ view that proving is important. I believe
this apparent inconsistency can be explained by Karis’ view of the purpose of proof as explanation. Karis believes that proof is important so that students can understand why things are true, and she seems to view her role as teacher as the one who ensures that these explanations are accessible to her students.

Some arguments were left without warrants in Karis’ classroom. Since I conjectured above that Karis provided warrants because it was important to her that her students have explanations to help them understand, it is reasonable to ask why she would allow some arguments to go without warrant, and under what circumstances this occurred. Karis left the warrants implicit in 24 out of the 75 arguments. Examining the context and mathematical content of each of these arguments resulted in the following two kinds of arguments involving implicit warrants. The first type of argument addressed mathematical ideas that calculus students could reasonably be assumed to know and understand, such as solving a quadratic equation. The other kind of argument involved mathematical ideas that were relatively new to students, ideas that had been recently introduced to the students in this class. For each of the arguments involving new mathematical ideas, a warrant was given for a similar argument earlier in the same class period, and this warrant was such that it could have been used in the later argument as well as the earlier one. For instance, on February 28, Karis and her students were discussing problems related to the Mean Value Theorem. In the first problem they discussed, the “requirements for the Mean Value Theorem” (Karis Calculus Class, February 28, line 8) were enumerated, that “it’s continuous on the closed interval and we can take the derivative on the entire open interval” (Karis Calculus Class, February 28, lines 10-11). This was the warrant for an argument that involved a claim that the Mean Value Theorem can be applied to $f(x) = x(x^2 - x - 2)$ on $I = [-1,1]$ and data that $f(x) = x(x^2 - x - 2)$ is continuous and differentiable. Later in the same
class, an argument involved a claim that the Mean Value Theorem applied to 

\[ f(x) = 2\sin x + \sin 2x \text{ on } I = [0, \pi]. \]

The data given was that the sine function is continuous and differentiable, and no warrant was explicitly stated. It seems reasonable that Karis may not have provided a warrant here because she expected her students to remember the requirements of the Mean Value Theorem from the previous problem. Karis provided warrants for arguments that involved mathematical ideas for which she thought her students needed explanations in order for them to understand, and she left warrants implicit when she did not see a need for explanation.

The types of warrants that were observed in Karis’ class included theorems, definitions, rule or procedures, and explanations of theorems, definitions, and procedures. In general, Karis’ warrants seemed to have the intention of explaining something to her students, whether it was why something worked the way it did or how to do a particular kind of problem. Most warrants collaboratively provided by Karis and her students were explanatory in nature, with Karis expanding upon students’ explanations as necessary. This theme of explanation was consistent throughout Karis’ actions with respect to proof and argumentation.

As Karis orchestrated and supported the argumentation in her classroom, she asked questions to elicit claims, usually suggesting what the claim should address. She also asked questions that prompted students to provide data for the claims. She added to or prompted other students to expand on claims, data, and warrants with which she seemingly was not satisfied. Her most visible support for argumentation consisted of providing explanatory warrants for arguments within her classroom. This focus on providing those explanations within argumentation seems to align with Karis’ conception of the purpose of proof as explanation. Karis saw justifying and proving as providing appropriate explanations, and she attempted to provide these appropriate explanations as warrants within her support for argumentation.
Jared: There Really Isn’t Much to Prove in Algebra

*Importance of proving.* Jared considered proof to be a topic of lesser importance in mathematics, but he acknowledged that it is sometimes helpful in remembering how to do things. In his first interview, I asked, “How important do you think proving is in math?” (Jared Interview 1, lines 30-31). He gave the following reply, speaking from the perspective of a high school teacher.

Depends on the type of students. I think that in lower level classes, no, it’s not very important. You just need to be able to get them to use the generalizations, theorems, whatever. But, in an honors class, in a calculus class, even a trig class, … in cases like that, proving is good. But, in classes where kids still can’t take a negative times a negative and get a positive number, what’s the point? To me, that’s what I feel we need to spend more time getting the students to be able to do the math, do the skills. (Jared Interview 1, lines 32-41)

This view of proof as not particularly important in school mathematics, especially for some students, is consistent with those expressed by some of Knuth’s (2002b) participants, who viewed proof as “an appropriate idea only for those students enrolled in advanced mathematics classes” (p. 73).

I asked Jared how important proving was to him, as he was learning math. His reply indicates a focus on the possible utility of proofs in remembering how to do mathematics.

Honestly I’m like why do I need to know this if I don’t need to know it for the test. But, as I look back, I think I appreciated a few ideas more because I knew where it came from. Like at the time it was like this is so stupid, but now, like looking back, it’s like oh, that does make sense now, and I do understand that more, and I do remember how to do this because we did the proof. (Jared Interview 1, lines 44-51)

His response regarding his own learning echoes that of one of Mingus and Grassl’s (1999) participants, who said, “It is easier to remember and understand how to do math if you know why it works and explore why it works for yourself instead of having someone just tell you how to do it” (p. 441). This view of proof as important in assisting with remembering how to do
particular procedures contrasts with Karis’ focus on the explanatory purpose of proof as important for student understanding of underlying mathematical ideas.

**Proving and procedures.** Jared tended to talk about mathematics as a procedural subject. As mentioned above, Jared prioritized students being able to “do the math, do the skills” (Jared Interview 1, line 51) over their being able to prove mathematical statements. This may have been partially because Jared viewed algebra as a subject in which there was very little to prove. In his second interview, he stated, “There’s not really much to prove in algebra” (Jared Interview 2, line 129). Jared seemed to view proving itself as somewhat of a procedural act. In particular, when asked to prove that the sum of the first n natural numbers is \( \frac{n(n+1)}{2} \), Jared used a procedure that resembled proof by mathematical induction without including the crucial induction hypothesis. Figure 5 shows the written work from his argument, which is transcribed below.

J: Okay, so first it’s to prove the base case, which is \( n \) is equal to one, so one equals one plus one, so one is equal to one, so the base case is true. Then we have to assume it’s true for the base case and prove it for all \( n \) plus one.

I: Okay.

J: So we have to substitute \( n \) plus one in for \( n \). And simplify that [writes \( \frac{n+1}{2} \), which is now crossed out]. So it would be \( n \) plus one times \( n \) plus two over two, which um, [long pause]. If we say like here, like this is \( n \) plus one, I forget, um, I’m trying to think, like here if we substitute this is like \( n \) plus one equal to \( n \), it’s like \( n \) \( n \) plus one over two which is the same thing [writes (and then crosses out) \( \frac{n(n+1)}{2} \)].

I: Okay.

J: All right, sort of now.

I: I’m not sure what you were doing here [points at \( \frac{n(n+1)}{2} \), which is now crossed out].
J: Cause I say like, I don’t know what I’m doing. Well, um, so we show that this, I don’t know what I’m doing.

I: How would you write out this part of the statement? [points at “The sum of the first $n$ natural numbers is”]

J writes $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

I: Okay.

J: From one to $n$, or $i$ is equal to $\frac{n(n+1)}{2}$, so that he has $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, so then we do $\sum_{i=1}^{n+1} i$.

I: Okay.

J: Now I’m good now, I think [writes $\frac{(n+1)(n+2)}{2}$] so that he has $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$. And that’s equal to, I don’t know what I’m doing. And that’s equal to this [points at $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$] plus $n$ plus one [writes $+n+1$ so that he has $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} + n + 1$].

I: Okay. And how do you know that this [points at $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$] is equal to this [points at $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} + n + 1$]?

J: When you add it.

I: Okay, and you’re sure it comes out?

J: Um, let me try. Yes, because we can multiply this out, we have $n$ squared plus three $n$ plus two over two [writes $\frac{n^2 + 3n + 2}{2}$]. If we do this we need a common denominator, um, so two $n$ plus two [writes $\frac{2n+2}{2}$] over top of $n+1$, so that line reads $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} + \frac{2n+2}{2}$], so, yeah, $n$ squared plus $n$ plus two $n$ plus two [writes $n^2 + n + 2n + 2$], so $n$ squared plus three $n$ plus two [writes $\frac{n^2 + 3n + 2}{2}$].
I: [Long pause] How well do you think that argument would convince, say, a high school calculus student?

J: I think that I’d be satisfied with it as a high school student, I think. Because I’d be like oh, that’s cool that this is a sum plus \(n\) plus one is the same as taking this sum. Adding one more thing to this \(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\) is the same as this \(\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}\). That’s cool, that works, that works out like that.

(Jared, Interview 1, lines 161-233)

Figure 5. Written work from Jared’s argument that the sum of the first \(n\) natural numbers is \(\frac{n(n+1)}{2}\).

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.
\]

\[\begin{align*}
\text{Base case} & : \quad n = 1 \\
& : \quad \frac{1(1+1)}{2} = 1 \\
& : \quad \checkmark
\end{align*}\]

\[\begin{align*}
\text{Assume true for } n \Rightarrow & : \quad \frac{(n+1)(n+2)}{2} = \frac{(n+1)(n+2)}{2} \\
& : \quad \frac{(n+1)(n+2)}{2} \\
& : \quad \frac{n(n+1)}{2}
\end{align*}\]

\[\begin{align*}
\text{Inductive step} & : \quad \frac{(n+1)^2 + 3(n+1) + 2}{2} \\
& : \quad \frac{n(n+1)}{2}
\end{align*}\]

It seems clear from Jared’s work that he was attempting a proof by induction, but he approached this in a procedural manner without reference to what it means to prove by mathematical induction. He knew the first step was to establish the truth of the base case, which he did. He also knew he needed to do something with the case “\(n + 1\).” However, the logic that he expressed is summed up in his last statement, that a high school student could understand it.
because he or she would understand that adding another term to \( \frac{n(n + 1)}{2} \) is the same as \( \frac{(n + 1)(n + 2)}{2} \). From his expressed logic, it is not clear that he understood the underlying logic of a proof by mathematical induction, even though from his work it is clear that he had previously encountered proof by mathematical induction. For him, following the steps was the key to completing the proof.

Supporting argumentation. Jared’s support for argumentation reflected his procedural emphasis and his belief that doing mathematics, at least for his students, involved doing skills rather than emphasizing proofs. Jared supported argumentation by providing warrants (he provided slightly more than half of the warrants in his class) and prompting or providing claims and data within episodes of argumentation.

In Jared’s class, students participated in generating most of the claims and providing most of the data. A typical episode of argumentation in Jared’s class tended to include a mathematical problem or expression being written on the board, students giving an answer to the question Jared asked (the claim), and some combination of students and Jared giving the data and warrant for that claim. On March 20, Jared asked his students to find the degrees of three polynomials he had written on the board. After they had worked for a few minutes, he asked them to share their answers. The following dialogue concerned the polynomial \( 14x^2y^6 - 3x^3y^4 \).

\[
J: \text{And then the third one, we have fourteen } x \text{ squared } y \text{ to the sixth minus three } x \text{ cubed } y \text{ to the fourth. [He calls on a student.]}
\]

\[
S: \text{Eight}
\]

\[
J: \text{Eight, how did you find the degree of the polynomial?}
\]

\[
S: \text{Added the exponents of } x \text{ and } y
\]

\[
J: \text{Okay, what did you get for this one [points at } 14x^2y^6]?
\]

\[
S: \text{Eight}
\]
J: Okay, this one [points at the next term].
S: Seven
J: So you know that eight is
S: Bigger
J: Bigger than seven, good. So all we’re doing, we’re just finding the degree of each of the individual terms, then whichever has the highest degree, that represents the degree of the polynomial.

(Jared Algebra Class, March 20, lines 248-261)

As can be seen in Figure 6, a student makes a claim that the degree of $14x^2y^6 - 3x^3y^4$ is eight. Jared asks the student for data, and the student says, “Added the exponents of $x$ and $y$” (Jared Algebra Class, March 20, line 252). Jared prompts the student to clarify the data, and the student gives the degree of each of the terms and states (with prompting from Jared) that eight is “bigger” (line 258) than seven. Thus the student makes the claim, and Jared and the student together provide the data. Jared gives the warrant by reiterating the procedure that was just illustrated, “So all we’re doing, we’re just finding the degree of each of the individual terms, then whichever has the highest degree, that represents the degree of the polynomial” (lines 259-261).

Figure 6. Diagram of typical argument in Jared’s class; denotes a student contribution, denotes a teacher contribution, and denotes that both contributed.
Importance of warrants. The warrants Jared provided tended to be procedures or rules; sometimes they were labeled as such, as when he said, “So, based on our rules, we know that seven is greater than five or two so seven is the degree of this polynomial” (Jared Algebra Class, March 20, lines 234-235, emphasis added) in response to a student’s giving the degree of a polynomial that was written on the board. At times Jared’s warrants included a statement that he treated as definitions, as when he said, “Degree is the sum of the powers of the variables. There are no variables present, so we have nothing to add, so it’s zero” (Jared Algebra Class, March 20, lines 174-175, emphasis added) in response to a student’s answer of zero to the question of the degree of the number twelve.

When students provided warrants in Jared’s class, it was generally in response to a question from Jared such as, “What do we need to do?” (Jared Algebra Class, February 21, lines 68-69). This question was regularly asked after a claim (in the form of a number or a symbolic expression) had been stated, and data (in the form of some other information about the problem) had been provided. Jared’s questions resulted in warrants from students that were also primarily rules and procedures. For instance, in response to “What do we need to do?” on February 21 in the context of finding the slope-intercept form of a line, a student replied, “Subtract x from both sides” (Jared Algebra Class, February 21, line 70), and in response to the same question on March 28 in the context of multiplying a binomial and a trinomial, a student said, “Distribute” (Jared Algebra Class, March 28, line 289). In each case, the student supplied an answer that indicated a procedure to be followed.

The kinds of warrants Jared provided and encouraged his students to provide parallel his view of the purpose of proof as a way to remember how to do mathematical procedures. Jared revealed a procedural orientation that was consistent across his own work with proving and
validating arguments and his support for collective argumentation within his classroom. This procedural orientation was probably more evident in this classroom because he saw very little need for proof in algebra, and his view of his students included a view that they could be characterized as lower level students. He saw proving as important for those students who were in the more advanced classes, but not for students in a class like the one he was teaching.

Relating Karis’ and Jared’s Conceptions of Proof and Support for Argumentation

Despite their dissimilar views of proof, there were some similarities between Karis and Jared’s support for argumentation. Both student teachers participated in most episodes of argumentation; each of their main supports for argumentation involved providing warrants; both asked questions that prompted students to give claims, data, and sometimes warrants; and the pattern of conversation in each of their classes was similar. The differences between Karis and Jared’s support for argumentation included the primary types of warrants each contributed; the order in which parts of an argument were contributed in their classes; and whether the discourse was calculational or conceptual (as used by Cobb, 2002). Within the similar structures of Karis’ and Jared’s support for argumentation are differences that resemble differences in their conceptions of proof.

Order of argument elements. The pattern of discourse in both student teachers’ classes basically followed the IRE model (as described by Mehan, 1979), in which the teacher initiates [I] by asking a question or providing a prompt, one or more students respond [R], and the teacher evaluates the response [E] before or while initiating another exchange. Within this pattern of discourse, both Karis and Jared asked questions that prompted students to offer claims, provide data, and sometimes to contribute warrants. However, the order in which these elements of argumentation were requested differed in the two classrooms. In Jared’s class, students usually
made a claim before the data and warrant were requested or provided, while in Karis’ class, often the data and warrant were contributed before the claim was completed. Jared generally asked for the answer, and then asked a student how he or she obtained the answer. Jared’s focus on the answer in class parallels his focus on one right answer when it came to constructing a proof—he focused on the end rather than the process. Karis’ conception of proof emphasized explanation and understanding. The order in which arguments were constructed in her class parallels this emphasis as she implicitly deemphasized the answers in favor of the explanations.

_**Conceptual and calculational discourse.** Jared emphasized not only the answer, but also how to get the right answer. In his class, the conversation tended to be procedural in nature, focusing on how to do the mathematical skill that was the topic of the day. This discourse can be characterized as calculational (as used by Cobb, 2002), focusing on producing results. In contrast to this, discourse in Karis’ class was sometimes calculational and sometimes conceptual. That is, Karis and her students tended to talk about the reasons behind the calculations they made rather than focusing exclusively on the calculations themselves. There were times that Karis and her students did focus exclusively on the calculations, but Karis’ warrants tended to articulate the reasons behind the mathematical processes in which her students were engaged. Part of the reason for this difference may be attributed to the different classes that Jared and Karis were teaching, particularly since Jared’s view of algebra included a perception that “there is not much to prove in algebra” (Jared Interview 2, lines 129-130). However, this distinction parallels the difference between Karis and Jared’s conceptions of the purpose of proof in mathematics. Jared believed that proof primarily helps one to understand how to do the mathematics, while Karis believed that proof was important to explain the underlying mathematical ideas to her students.
Different uses of warrants. This distinction between calculational and conceptual discourse also captures some of the differences in Jared and Karis’ use of warrants to support collective argumentation. Both Karis and Jared used procedures as warrants in their support for collective argumentation. However Karis’ procedures-as-warrants differed from Jared’s procedures-as-warrants. Many of Jared’s procedures-as-warrants were simple phrases, such as “add like terms” or “subtract x from both sides and divide by negative one” that were used frequently by both Jared and his students. Karis’ procedures-as-warrants tended to be longer and of greater complexity. Her procedures often seemed to function as both a warrant for the specific argument and a general guideline for solving problems similar to the one in the argument. For instance, in an argument regarding a collapsing sum, Karis said, “I write out the terms, as many as I need to, to see the pattern, cross off what cancels, and then just simplify it” (Karis Calculus Class, March 29, lines 88-89). In this warrant, Karis is providing students with a procedure for simplifying a collapsing sum. She makes clear how she makes a decision regarding how many terms to write out, and, as she does so, provides an explanation for her students as to what they should expect as they write out the terms and find ones that cancel—they should look for a pattern. This type of procedure resembles Star’s (2005) description of a heuristic procedure that “indicates quite sophisticated and deep knowledge” (p. 407) as opposed to a procedure that is an algorithm, as one might characterize Jared’s procedures-as-warrants.

Importance of warrants. Yackel (2002) described “recognizing the importance of warrants and backings” as part of the teacher’s role in argumentation (p. 439). I do not argue that Karis and Jared recognized the importance of warrants within the collective argumentation that occurred in their classrooms. However, the types, frequency, and timing of warrants differed in their classrooms in ways that can be linked to their conceptions of the purpose and need for proof
in mathematics. The nature of their prompting and provision of warrants was important in their classes despite their lack of explicit knowledge of argumentation structures or specific concern with argumentation. There is no evidence that Jared and Karis were focused on supporting argumentation as an explicit goal of their instruction, nor did they have a goal of examining the argumentation of their students. In many previous studies involving argumentation, the teacher has been the researcher or the instruction was intentionally planned to foster collective argumentation within the classroom (e.g., Osborne, Erduran, & Simon, 2004; Whitenack & Knipping, 2002; Yackel, 2002). Despite the differences in context between the studies focused on developing students’ collective argumentation and the current study focused on the nature of argument as it arises naturally in student teachers’ classrooms, the use of warrants was seen in both contexts as a critical aspect of the teacher’s role in the support of argumentation in the classroom.

*Structure of arguments.* Forman and Ansell (2002) and Yackel describe the critical role of the teacher in structuring collective argumentation. Jared and Karis each structured the argumentation in their classrooms by asking questions that prompted students to make claims or provide data or warrants and by providing claims, data, and warrants themselves. This structure is different from what is described by Cobb (1999) or McCrone (2005) who wrote about facilitating discussions by making decisions about who should contribute solutions at what times based on observations of small group work. Because Karis and Jared did not tend to use small group work in their classes, this particular aspect of facilitating discussions was not observed in their classrooms. However, Karis and Jared did structure the argumentation in their classes by asking questions that elicited claims, data, and warrants in a particular order from their students.
Conclusion and Implications

The differences in how these student teachers support argumentation in their classrooms aligns with similar differences in their conceptions of proof, particularly in their conceptions of the purpose and need for proof in mathematics. This study provides an empirically grounded description of the relevance of this aspect of a conception of proof to argumentation as a particular aspect of classroom practice of prospective secondary mathematics teachers. The extent to which these student teachers’ support for argumentation aligns with their conceptions of proof suggests that their conceptions of proof may influence the way in which they support collective argumentation in their classrooms. Investigating particular ways in which this influence occurs would provide fruitful ground for future research.

The description of practice in regular high school classrooms in which the support of argumentation was not an explicit goal suggests that argumentation is present to some extent in many classrooms. It may not appear in forms that are as rich as the argumentation described in studies where its facilitation was a goal of instruction. The presence of this argumentation suggests a base upon which more robust argumentation practices may be built. Examining argumentation is one way to look at an important aspect of classroom practice, and the modification of Toulmin’s diagrams provides a natural way to look at the structure and patterns within the argumentation.

An evidence-supported link between a prospective teacher’s conception of the purpose and need for proof in mathematics and his or her support for collective argumentation has implications for teacher educators and for all those who work with mathematics teachers. In particular, this study should serve to sensitize those involved in the mathematical and pedagogical preparation of secondary mathematics teachers to some of the issues involved in
relating what secondary mathematics teachers have learned in mathematics and mathematics education classes to how they facilitate argumentation in secondary classrooms. While proof is often central to undergraduate mathematics classes, the focus in these classes seems to be on learning to prove, and some students in these classes may focus on being able to prove or produce proofs for a test. This study suggests that an important additional focus for mathematics and mathematics education courses for prospective secondary mathematics teachers is not only how to prove but how and why proving is important within mathematics. A focus on the role of proof in mathematics is important because it may influence how a future teacher will support argumentation with his or her students. This study may also serve to inform those who work with teachers of a possible explanatory factor for observed patterns of argumentation in secondary mathematics classrooms and suggests that an appropriate focus of professional development may be the purpose and role of proof in school mathematics.

While this study provides preliminary evidence to connect one aspect of a conception of proof to one aspect of collective argumentation, much future work remains to examine other aspects of a relationship between proof and argumentation, particularly with regard to student learning. Examining carefully other aspects of this relationship, in settings involving both prospective and practicing teachers, remains an important goal. In addition, future research should consider which perceptions of the purpose and role of proof in mathematics are more productive in facilitating classroom argumentation, perhaps by identifying and studying teachers who facilitate argumentation in ways that support student learning.
References


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