In an epistemology where mathematics teaching is viewed as goal-directed interactive communication in a consensual domain of experience, mathematics learning is viewed as reflective abstraction in the context of scheme theory. In this view, mathematical knowledge is understood as coordinated schemes of action and operation. Consequently, research methodology has to be designed as a flexible, investigative tool.

The constructivist teaching experiment is a technique that was designed to investigate children’s mathematical knowledge and how it might be learned in the context of mathematics teaching (Cobb & Steffe, 1983; Hunting, 1983; Steffe, 1984). In a teaching experiment, the role of the researcher changes from an observer who intends to establish objective scientific facts to an actor who intends to construct models that are relative to his or her own actions.

I. ROLES OF THE RESEARCHER IN A TEACHING EXPERIMENT

A distinguishing characteristic of the technique is that the researcher acts as teacher. Being a participant in interactive communication with a child is necessary because there is no intention to investigate teaching a predetermined or accepted way of operating. The current interest always lies in hypothesizing what the child might learn and finding ways and means of fostering this learning. Based on current interpretation of the child’s language and actions, the experimenter makes decisions concerning situations to create, critical questions to ask, and the types of learning to encourage. These on-the-spot decisions represent a major modus operandi in teaching experiments and the researcher has the responsibility for making them.

Beyond acting as teacher, another role of the researcher is to analyze the knowledge involved in teaching. The researcher must build what Hawkins (1973) called a map and what I call a model of each child’s mathematical knowledge. Toward this end, the teaching experiment is primarily an exploratory tool, derived from Piaget’s clinical interview and aimed at investigating what might go on in children’s heads. This process involves formulating and testing hypotheses about various aspects of the child’s goal-directed mathematical activity in order to learn what the child’s mathematical

knowledge might be like. It is my belief that the researcher can best formulate and test hypotheses and interpret the results of the tests in intense interactive communication with the child, so that a close personal and trusting relationship can be formed. The formulation and tests of hypotheses involve initiating probes that might stretch the child to the limits of his or her conceptual adaptability and endurance. So, the researcher must strive to develop an operating context in which the social meanings of the involved language and actions are negotiated by the participants (Cobb & Steffe 1983, pp. 84ff). Above all, the activity of doing mathematics should be viewed with a playful attitude and confidence. To promote a playful attitude, the researcher might, for example, develop the expectation in a child that he or she can create situations for the teacher to solve. Such situations can not only reveal the child’s current level of development but they also tend to encourage the child to develop the confidence occasionally to control the interactive communication and to make decisions concerning the situations he or she wishes to explore.

Finally, because the teaching experiment involves experimentation with the ways and means of influencing children’s knowledge, it is more than a clinical interview. It is directed toward understanding the progress children make over extended periods of time, and one of the main goals is to formulate a model of learning the particular content involved. Consequently, each teaching episode is video taped and the recorded material is retrospectively reviewed after the teaching experiment. Along with the experiences of the researcher that led up to this review, models of the constructive activity of the children over the duration of the teaching experiment are formulated (Steffe, Cobb, & von Glasersfeld, 1988). It is crucial to understand the modeling process as an organizing activity analogous to what Treffers calls mathematizing (Treffers, 1987, p. 51). The basic and unrelenting goal of a teaching experiment is for the researcher to learn the mathematical knowledge of the involved children and how they construct it. In the following sections, I intend to illustrate what I mean by learning the mathematical knowledge of children and to make explicit some of its implications for mathematics education.

II. MATHEMATICS OF CHILDREN

One of the fundamental results of Piaget’s genetic epistemology (Piaget, 1970) is that the roots of mathematical knowledge can be found in the general coordination of the child’s actions. From a constructivist point of view, “the essential way of knowing the real world is not directly through our senses, but first and foremost through our material or mental actions” (Sinclair, 1990). There are individual actions like throwing, pushing, lifting, and the like. Mathematical knowledge is based on coordinations of such actions into organized action patterns to achieve some goal. I take these goal-directed action
patterns to be what Piaget meant by sensory-motor schemes (Piaget, 1980b).² The mathematical knowledge of children can be understood in terms of goal-directed action patterns if “action” is taken to refer to mental as well as to physical action. These mental actions constitute operations – interiorized action patterns – and the involved schemes are operative rather than sensory-motor. From the perspective of how school mathematics is viewed today, this perspective on children’s mathematical knowledge is revolutionary, and, if accepted, would transform mathematics education in a fundamental way.

To illustrate what I mean by “operative scheme”, let us take how an eight-year-old child, Maya, understood division and multiplication. Maya was a participant in my most recent teaching experiment (Steffe & von Glasersfeld, 1985). When it is the goal to specify and to chart modifications of schemes, it is often advantageous to teach individual children. Choosing to work in these laboratory conditions should not be construed to mean that I view constructivism as being restricted in its implications to teaching individual children. Even in that case, interactive mathematical communication is the foundation of the teaching experiment. This is illustrated in the following protocols which document how I interpreted Maya’s language and actions as well as the subsequent decisions made. The results of implementing the decisions are also documented and a qualitative analysis of what Maya learned is provided. The protocols demonstrate how essential social interaction is in cognitive construction (Piaget, 1964; von Glasersfeld, 1990).

2.1 Maya’s Schemes for Dividing and Multiplying

In one of the teaching episodes with Maya, I presented her with 21 numeral cards in a row and then hid them from her view. “T” is used in the protocol to indicate my language and actions and “M” is used to indicate Maya’s.

T: If you start from there (the beginning of the covered row) and take three cards at a time to make a pile, I wonder how many piles of three you could make? M: (Sits silently in deep concentration for approximately two minutes) Seven! W(Witness): When you counted, what did you say? M: 21, 20, 19 – that would be one; 18, 17, 16 – that would be two; etc.

Maya’s method for dividing by three was to count backward by one, take each trio of number words as a unit, and count those units of three as she made them. This method was not suggested to her by me and differed substantially from the computational algorithm her classroom teacher had tried to teach her. It was an operative scheme she independently used to solve what adults commonly call “quotative division problems”.

It is too strong to claim that Maya’s material counting actions constituted her concept of division because of the way she organized counting by one into units of three. She intentionally counted to reach a goal. So, I looked further for the operations Maya might have used in assimilation² of the task. The available evidence was that Maya seemed to use her counting scheme and her concept of
“three”. My hypothesis was that her concept of division was quite different from what adults call “quotative division”.

As a concept, quotative division would have involved an awareness of the result of making, say, units of three using the 21 individual units, a unit containing units of three, and a method for finding the numerosity of the containing unit before counting. Because Maya seemed to be aware of the units of three that she made as she counted backward, my question was whether she was aware of the containing unit before she started counting. To investigate this question, I asked Maya to formulate the results of her dividing scheme into a multiplication problem. If she was aware of the structure of her results of counting, that would be an indication she had a concept of quotative division.


“I am figuring out how many threes equals seven” indicates to me that Maya was solving a new problem while she was sitting silently. She did not seem to use the units of three she made when counting backward by one as material for further operating. Why she did not take the result, seven, as how many units of three she had made, might be understood if one deceners and tries to assume Maya’s point of view.

2.2 Maya’s Concepts of Multiplication and Division

Maya had a result of counting, seven, that achieved her goal. To then take seven as a new given and formulate a multiplication problem would involve her becoming aware of how she had arrived at seven. She would have to represent3 the activity of counting and isolate the structure of the activity in its represented result. In other words, without any perceptual records of counting available, she would have to combine the units of three that she made and take seven as the given numerosity of the resulting unit containing those units of three. This seemed to demand an abstraction of the structure of operating that I hypothesized Maya had not made. To test this hypothesis, I investigated if she could additively combine six units of three and five units of three. To begin, I asked her to count out eighteen blocks by three.

T: Put eighteen blocks into that container. You can count them by three if you want. M: (Maya takes three blocks at a time and places them into the container). T: Give me a multiplication problem for that. M: (Long pause) three times six equals eighteen!

During the long pause, Maya apparently re-enacted counting to eighteen by three, silently recording how many times she made a unit of three—“1–2–3, that would be one”, etc. She used the result of this activity to formulate her sentence “three times six equals eighteen” because she did not keep records of
how many perceptual units of three blocks she made nor did she know that “3x6 = 18” from her regular classroom work. Continuing, I then asked Maya to put fifteen blocks into another container. Maya put them in by three and this time she kept track and said that she had five groups of three in the container. I then poured the contents of the two containers together and asked Maya to find the number of blocks in that combined collection of blocks using her units of three. The only answer Maya could give was “thirty” (Maya knew “five times six is thirty”).

Her failure to additively combine the two lots of three is an excellent indicator of the nature of her concept of multiplication. She simply did not reason with units of three as she could reason with units of one. Her concept of multiplication consisted in anticipating using a unit of three to segment counting by one; and actually segmenting counting by one into trios a given number of times constituted her multiplying actions. This concept of multiplication is less powerful than repeated addition because in that case the operations used involve reasoning with composite units rather than with units of one.

Maya’s language — “take three out of twenty-one” — indicates that her concept of division did involve extracting three from 21 — a part-whole operation. But she did not seem to understand that the results of dividing would be a unit partitioned into units of three. Rather, her concept of division consisted in anticipating segmenting her number sequence from 21 down to and including one into trios. She could anticipate making units of three but did not see the abstract structure of the result as do children who have constructed quotative division. Actually segmenting counting by one into trios, and counting the trios so formed, constituted her dividing actions.

I have called operative schemes, like Maya’s multiplying and dividing schemes, mathematics of children (Steffe, 1988a). While Maya’s schemes have epistemological significance in the study of the genesis of multiplicative structures (cf. Piaget, 1970, pp. 13 ff), they also have educational significance for mathematics education. As mathematics educators, we have a choice between using mathematics of children or conventional school mathematics as the basis on which to teach mathematics. Choosing the former is a fundamental requirement of constructivism for mathematics education.

III. MATHEMATICS FOR CHILDREN

The first situation I illustrated to establish Maya’s level of development in multiplication and division was not a problem for her. She didn’t have a problem if that means “to search consciously for some action appropriate to attain a clearly conceived, but not immediately attainable, aim” (Polya, 1962). In Maya’s assimilation of the division situation I posed, she apparently experienced an awareness of making more than one unit of three — an awareness of indefinite numerosity. This led to her making units containing three back-
ward counting acts and coordinating those units with her standard number word sequence to make her awareness of indefinite numerosity definite. Counting seemed to be a constitutive part of her division concept and was activated by her awareness of indefinite numerosity. In this sense, the response of her dividing scheme was already available and was not an object of a search. This is reminiscent of a comment made by Piaget (1964) in a critique of associationism: “A stimulus is a stimulus only to the extent that it is significant and it becomes significant only to the extent that there is a structure which permits its assimilation, a structure which can integrate this stimulus but which at the same time sets off the response” (p. 15).

In scheme theory, what it means to have a problem is, first, for a child to have a scheme to which the given situation can be assimilated and, second, for the scheme’s outcome to be inadequate to remove disequilibria that may have been created as a result of the assimilation. If the child has not associated the action appropriate to remove a disequilibrium with the assimilated situation that triggered the scheme, the child’s search for the action would be unsuccessful. This is precisely what happened when Maya tried to solve two of the tasks that I presented above. She seemed to have a problem when I gave her a multiplication task after she had used her division scheme and said “seven”. Moreover, she seemed to have a problem when I combined into one container the six groups and the five groups of blocks she had made, and asked her how many trios were in the container. To solve either of these two problems would have involved a reorganization of her multiplying and dividing schemes.

At the time, I searched very hard for situations in which Maya would make these reorganizations. My searches for actions to solve the problem I had were as unsuccessful as Maya’s searches for actions to solve the problems she had. In retrospect, I could find no reason to include inversion between multiplication and division and distributive reasoning in Maya’s zone of potential development (Vygotsky, 1956) at any time during her third grade in school. As mathematics educators, we have a choice between determining the mathematics for children through interactive communication or, on the other hand, through the conventional meaning of terms like “borrowing”, “quotative division”, or “distributive property”. Choosing the former is a second requirement of constructivism for mathematics education.

IV. TYPES OF LEARNING

The mathematics that children can learn is what I mean by mathematics for children. In the illustrations of mathematics that Maya could learn in the teaching experiment, I take learning as consisting in her modifications of the multiplying and dividing scheme explained above. For Piaget (1980b), assimilation is the fundamental relation involved in learning. It can involve or lead to perturbations, but the restoration of equilibrium that occurs as a result of
neutralizing a perturbation by using a scheme might not involve learning. For example, after Maya had reached “seven” when counting trios of backward counting acts, she experienced a result that made definite her indefinite awareness of “more than one” that drove counting. She had used her counting scheme, but there were no modifications in it that were observable to me and, hence, no learning. On the other hand, modifications of a scheme would not occur without perturbations.

**Functional accommodations.** I have isolated two basic types of accommodation – functional and metamorphic (Steffe, 1988b). A functional accommodation of a scheme is any modification of the scheme that occurs in the context of using it. A child can make a functional accommodation in a scheme by **hindsight** (Hartmann, 1942) – by viewing the results of using a scheme in a novel way. For example, given how Maya used her dividing scheme, learning the basic division facts was in her zone of potential development. An association between a division situation such as Maya experienced and the result of using the scheme could have been established by Maya had I encouraged her to reflect on her immediate past experience and to abstract the involved language (“twenty-one divided by three” and “seven”). This is reminiscent of the principle of association explained by Guthrie (1942): “A stimulus pattern that is acting at the time of a response will, if it recurs, tend to produce the response” (p. 23). Although I disagree with him that the principle of association is the basic event in learning, its reformulation in scheme theory is powerful enough to explain how children like Maya might form associations called “basic facts” as a result of what Piaget calls reflective abstraction (Piaget, 1980a).

V. PROCEDURAL FUNCTIONAL ACCOMMODATIONS

Functional accommodations that I call procedural are not restricted to hindsight. A child might also modify the activity of a scheme without curtailing it. For example, approximately one year after I had isolated Maya’s schemes for multiplying and dividing, I again investigated if distributive reasoning was in her zone of potential development. This investigation was promoted by the progress she had made in her concept of multiplication. It had become an iterative concept – “seven times twelve” now meant twelve seven times. So, it seemed to be plausible that she could learn to combine twelve four times and twelve three times, to produce twelve seven times, and to separate the latter to produce the former. She did modify the activity of iterating a composite unit in the following protocol.

T: (Places a strip of adding machine paper in front of Maya) this is 36 inches long. How many 12-inch rulers is it going to take to measure it? M: Three. T: How did you know? Did you know that three times twelve is thirty-six? M: (Shakes her head “yes”). T: (The 12-inch ruler was divided into a 10-inch and a 2-inch ruler) now, if you put the tens down first, how many of those would it take and how many of the twos? M: It would take three of these (the
10-inch ruler) and three of these (the 2-inch ruler). T: Tell me why. M: Because 3 tens is thirty. Two times three is six and thirty plus six would be thirty-six!

As her teacher, I was excited because it seemed possible that Maya had deeply reorganized her multiplication concept. If this proved to be the case, then Maya would have a solid basis for constructing standard multiplying algorithms as child-generated algorithms (cf. Steffe, 1983; Hatfield, 1976) because she would be able to produce these algorithms as functional modifications (accommodations) of her current operative multiplying scheme.

Another possibility was that Maya had merely modified the activity of multiplying rather than her concept of multiplication. We do see in the protocol that Maya explained why it would take three of each smaller ruler by finding each product and adding their results together. There didn’t seem to be the logical necessity that an understanding of distributivity would provide – there would be no reason to find the sub-products and add them together if distributivity was a property of her concept. However, she might have used distributivity to obtain the result and then used computing to explain why it worked.

A third possibility was that Maya had made a local modification in her multiplication concept as well as a modification in the activity of multiplying because, in the problem preceding the one of the protocol, I had presented her with a 24 inch strip of adding machine paper and the separated 12-inch ruler which, in turn, separated the adding machine paper into two congruent parts. Without moving the ruler, Maya immediately saw that it would take two 12-inch rulers and two of each smaller ruler to cover the strip of paper. She could easily anticipate putting another 12-inch ruler beside the one on the paper to cover the uncovered part, closing the potential measurement activity. The results of measuring thus formed a visual whole and the parts could be easily isolated and recombined with respect to the visual whole.

If the third interpretation proved to be the most plausible, the local modification of her multiplication concept would be restricted to situations similar to the one in which it occurred. To investigate if this was so, I presented her with an 84-inch piece of adding machine paper.

T: See if you can do this one – 84 inches (places an 84-inch strip of adding machine paper in front of Maya). Suppose you use the 10-inch one first and then the 2-inch one second. They have to be equal like they were before. How many of each kind would it take? M: (Long pause) do they have to be equal? T: Tell me one that isn’t. M: I don’t have one. T: If you find one, tell me. M: You could use eight of these (10-inch ruler) and two of these (2-inch ruler). T: Can you find two that are equal? M: (Long pause where Maya is in deep concentration). I can’t find any!

It would have required a deep insight for Maya to use her knowledge of the associated multiplication fact. Had she said “seven because seven times twelve is eighty-four”, I would have inferred that she understood that so many iterations of a 12-inch ruler is equivalent to the same number of iterations of a 10-inch and a 2-inch ruler. From this episode, I concluded that the third
interpretation was the most plausible. However, I still was not satisfied. Could this local modification lead to a deeper reorganization of her multiplication concept? So, I went on, encouraging Maya to experiment. She tried 9 upon my suggestion and found that it did not work because “nine times ten is ninety”. She then selected 7 as a possibility after I asked her what the next logical choice would be. After finding the two sub-products and adding their results, I asked her to tell me the result of multiplying seven and twelve.

T: By the way, what is seven times twelve? M: (Immediately) eighty-four. T: How did you know that? Did you just know it or did you use this to help you (pointing to the measurement situation)? M: No (indicating she did not use her measurement results). T: Could you use this to help you? Suppose you did not know seven times twelve, how could you use it? M: Seven twelves, you would put those two together (the 10-inch and the 2-inch ruler).

Maya clearly understood that if she placed the 12-inch ruler down seven times, it would measure 84 inches, just as would placing the 10-inch and the 2-inch ruler down seven times. However, she combined the 10-inch and the 2-inch rulers together to make the 12-inch ruler as an explanation for why the basic fact and the measurement led to the same result. Distributivity was still not a property of her reasoning before measuring.

I do believe that using products for which Maya had no established associative bonds held promise as a learning environment in which she could reorganize her concept of multiplication, because strategies of reasoning like the one displayed could be encouraged. Distributivity was now in her zone of potential development. But, I still could not infer that she had made a general reorganization of her concept of multiplication. She did modify her activity of multiplying in the measurement situation and there was also a local modification of her multiplication concept. But there was no general reorganization of multiplication that would be manifest in the flexibility made possible by distributive reasoning.

I would not have known this had I not generated possible interpretations (von Glasersfeld, 1987) of Maya’s modification of multiplying and tested which was the most plausible by investigating the generality of the modification. In the situation I used as a test, I simply varied the length of the strip of paper with the intention of creating a problem that Maya could solve with my help. Our interactive communication finally enabled me to interpret her modification. As mathematics educators, we have a choice between interpreting children’s mathematical activity in learning environments through interactive communication, making qualitative distinctions, or focusing on the result or product of their activity. Choosing the former is a third requirement of constructivism for mathematics education.
In the case of the 36-inch piece of adding machine paper, Maya did not have a problem. Her utterance “three” indicated an assimilating generalization – differences in the new situation were manifest in the results of using the operations of the assimilating scheme. Her initial response in the case of the 84-inch strip was also the result of assimilating generalization. But, when I said “find two that are equal”, she genuinely had a problem whose immediate solution would involve an insight as explained by Wertheimer (1959).

what a change, when ... all the parts suddenly form a consistent clear whole, in a new orientation, in strong reorganization and recentering, all fitting the structural requirements (p. 57).

If an insight had led to the distributive actions that solved the problem when I asked her to find two that were equal, it would be a midight7 (Gates, 1942). But, given Maya’s current iterative multiplicative concept, there was little chance of her producing such an insight as a midight, foresight, or hindsight. The modification in her multiplication concept was restricted to interpreting the multiplicand as so many tens and so many ones.

Other than her iterative multiplicative concept and the functional accommodations explained above, Maya made few modifications in the primitive multiplicative scheme I isolated while she was in the third grade. However, as a result of her regular classroom instruction, she had learned the multiplication facts and the paper and pencil computational algorithms for multiplying one and two digit numbers as explained by Thorndike (1922) more than 65 years ago. Her algorithms consisted of an organization of associative bonds that included the basic facts, carrying, etc. (Gates, 1942). Maya had not produced the computational algorithms as modifications of operative schemes, if for no other reason than that she lacked distributivity. Her computational algorithms were connected to, rather than contained in, her iterative multiplicative concept. As such, Maya had “learned” a new scheme for multiplying that was not a modification of her more primitive scheme. It is little wonder that Nesher (1986) has commented, “No one has succeeded in demonstrating that understanding improves algorithmic performance” (p. 16), because algorithms are viewed as procedures disembodied from the mathematics of children and are imposed upon them without regard to the nature of their schemes. As mathematics educators, we have a choice between taking assimilation as the fundamental relation involved in learning and learning as consisting in the modifications of schemes or, instead, taking association as the fundamental relation involved in learning and learning as consisting of establishing and organizing bonds into a system. Choosing the former is a fourth requirement of constructivism for mathematics education. In that case where the associations are established as a result of reflective abstraction, one can include association as a relation involved in learning – but not the fundamental one – because the
associations can function in place of the curtailed operations.

Metamorphic accommodation. Had Maya abstracted the structure of her distributive actions after finding that “seven” worked, it would have amounted to a reorganization of her concept as explained by Wertheimer even though it would not have been a foresight. Regardless of when it occurs in a problem solving episode, such an insight would be an example of a metamorphic accommodation because it would lead to a general reorganization of the involved concept. From the observer’s perspective, if Maya spontaneously used distributive reasoning as a strategy to rename products, then a general reorganization could be inferred.

Any accommodation that occurs independently and involves a general reorganization of a scheme is called “metamorphic accommodation”. Had Maya’s multiplication concept been an abstract concept on a par with the concept of quotative division as I have explained it, she could have produced distributive reasoning as an insight when she solved some problem. However, given her iterative concept, an interiorization of the iterative actions would have been necessary for her to substantially reorganize the concept. Interiorization is the most general form of abstraction.

It leads to the isolation of structure (form), pattern (coordination), and operations (actions) from experiential things and activities; an interiorized entity is purged of its sensory-motor material (in Steffe, Cobb & von Glasersfeld, 1988, p. 337).

Knowing what interiorization leads to does not explain its process. I have found the process to involve the use of operations in the recreation of an immediate past or possible experience (Steffe, 1988b). For example, when Maya explained, “Because 3 tens is thirty. Two times three is six and thirty plus six would be thirty-six!” I believe she used numerical operations to enact her possible measuring experience. These numerical operations are precisely the operations that are needed to purge measuring activity of its sensory-motor material. However, one or more such episodes was insufficient to complete a developmental metamorphic accommodation – a metamorphic accommodation that occurs as a global result of such specific experiential episodes (Steffe, 1988b). It involves interiorization and reorganization of specific operations, where the reorganization is manifest in independent use of the operations in solving problems that have, from the researcher’s perspective, analogous structure.

From the researcher’s perspective, the purpose for engaging children in goal-directed activity that includes problem solving is not simply the solution of specific problems. The primary reason is to encourage the interiorization and reorganization of the involved schemes as a result of the activity. These interiorized and reorganized schemes constitute operative mathematical concepts that are constructed by means of reflective abstraction. As mathematics educators, we can treat operative mathematical concepts in one of two ways: (1) as constructed by children as a result of goal-directed activity in
learning environments, or (2) a priori to mathematics learning and disembodied from learning — as what the children are to learn this school year, this week, or today regardless of whether or how the children might learn them. Choosing the former is a fifth requirement of constructivism for mathematics education.

VII. CHILDREN'S MATHEMATICAL KNOWLEDGE AS TEACHER KNOWLEDGE

Maya’s schemes were formulated in my attempts to understand her ways and means of solving the situations I posed. Specifying children’s schemes does involve creating and posing situations that might reveal those schemes from their sensory-motor to their operative forms. But I do not focus on the situations per se. Rather, I focus on children’s solution attempts in a search for the schemes they might use in assimilation and solution. If I am able to specify the schemes that explain what children do, I can then create and pose new situations with the intention of isolating any modification they might make. Using this kind of a strategy, it is possible to learn the mathematical knowledge of the children I am charged with teaching.

It is not my intention to argue that my explanation of Maya’s operative schemes and their modifications are identical to Maya’s knowledge in this content area (cf. von Glasersfeld, 1983). Rather, my argument is that Maya’s language and actions were the experiences available to me in a learning environment as I strove to learn her schemes of operating and that my interpretation could be made only in terms of my own knowledge. So, from my point of view, my description of Maya’s mathematical knowledge is unavoidably an expression of my own concepts and operations. Although I do acknowledge that Maya had a mathematical reality separate from my own, I had no direct access to it.

Children’s mathematical knowledge, then, refers to our models of the mathematical knowledge of children. Learning children’s mathematical knowledge is part of what it means to be a mathematics teacher and no one else can assume this responsibility for the teacher. As mathematics educators, we can take the responsibility for learning children’s mathematical knowledge or we can take it as a given, similar to other parts of our own mathematical knowledge. Choosing the former is the sixth requirement of constructivism for mathematics education.

VIII. TEACHING AS GOAL DIRECTED ACTIVITY

A relativistic world view legitimizes children’s mathematical knowledge. To view students as important others who have a mathematics that is indeed worth knowing represents a general change of paradigm. The focus is shifted from the
teacher, not to the students, but to the interactive communication that transpires between teachers and students, among students, or among teachers (von Foerster, 1984; von Glasersfeld, 1988; Bauersfeld, 1988). Interactive communication constitutes the central activity of teaching and this activity is goal-directed in a way that is analogous to children's mathematical activity as I explained it. Mathematics educators may view mathematics teaching in this way or as the transmission of knowledge (McKnight, et al. 1987). Making the first choice is the seventh requirement of constructivism for mathematics education.

The nonrelativistic view of teaching as the transmission of knowledge is based on the idea that students must come to know mathematics as it is, a view that ignores the mathematical reality of children. It is a "missing child" paradigm and I wonder whether its advocates would consider the mathematics of children as what it is, not what the teachers wish it to be, as they do abstract mathematics. Constructivist teachers do not deny that children have mathematical realities distinct from their own. To the contrary, they actively seek to learn and to modify those realities. The teachers' models are not simply what they might wish them to be, for they are always subject to modification in interactive communication. What children do constrains teachers and constitutes a critical point of contact with the actual mathematical knowledge of children.

IX. LEARNING ENVIRONMENTS

The result of an assimilation of a particular situation is an experience of the situation and this experience constitutes a learning environment in the immediate here and now. From a teacher's perspective, a child's learning environment consists of what the teacher intends for the child to learn after the child assimilates a particular situation. To distinguish between the teacher's perspective of a child's learning environment and the child's perspective, "possible learning environment" and simply "learning environment" are used to refer to the teacher's frame of reference and to the child's frame of reference, respectively. Possible learning environments are determined by a teacher in part with respect to children's zones of potential development (Vygotsky, 1956) and in part with respect to children's actual level of development (Sinclair, 1988). The latter is determined by the situations the children can independently solve and the former is determined by the problems the children can solve with the help of a teacher. In the constructivist view, a zone of potential development relative to a child's specific scheme is determined by the modifications of the scheme the child might make in, or as a result of, interactive communication in learning environments. Like possible learning environments, a zone of potential development is the teacher's concept.

Because zones of potential development and possible learning environments depend on each other, it is possible for a child to have differing zones of potential development with respect to the same scheme. The particular modifica-
tions of a scheme could diverge in one of several directions depending on the possible learning environments encountered by the child which, in turn, are dependent on particular modifications. This realization places the teacher in a crucial position with respect to the mathematical development of his or her students. Nevertheless, the unavoidable dependence does not mean that a child's zone of potential development is wholly dependent on possible learning environments, nor does it mean that the child can learn anything the teacher wants the child to learn in the immediate future.

A particular modification of a scheme cannot be caused by a teacher any more than nutriments can cause plants to grow—nutriments are used by the plants for growth but they do not cause plant growth. Teachers are constrained in specifying what they place in a child's zone of potential development by what the child makes from their experiences in particular learning environments. Zones of potential development are negotiated, then, through the interactive communication that transpires in learning environments.

As a concept, a learning environment can be thought of as a variable experiential field whose contents are specified by its participants. As mathematics educators, we have a choice between viewing learning environments in that way or taking the situations of learning as being a priori and invariant throughout their use by children—considering our frame of reference as the only one. Choosing the former is an eighth requirement of constructivism for mathematics education.

X. TEACHING TO LEARN

Piaget (1964) identified experience as one crucial factor of intellectual development and this applies to us as mathematics educators as well as to our students. Mathematics educators who aim at learning children's mathematical knowledge and its evolution must assume that there will be some regularity in their experiential world. These recurring patterns are the basis for forming concepts (or models) of the developing mathematical knowledge of children. In fact, the term "model" refers to a conceptual generalization that is abstracted from a group of experiences for the purpose of categorizing and systematizing new experiences (von Glasersfeld & Steffe, 1987). I believe that mathematics educators can reach a point where they may feel justified in making predictions about further experiences of the child or about the experiences of children like those with whom they are currently working.

The generative power of children can be impressive when they are working in learning environments that are conducive to constructive activity (Steffe, Cobb, & von Glasersfeld, 1988). However, children's mathematical knowledge has only begun to be charted. Mathematics educators at all levels have an exciting choice between being participants in specifying the generative mathematical power of children or taking what children can learn as being
already specified by an a priori curriculum. Choosing the former is a ninth requirement of constructivism for mathematics education.

For mathematics educators who opt for constructivism, the following ten principal goals might serve as a guide in their research activities.

1. To learn how to communicate mathematically with students.
2. To learn how to engage students in goal-directed mathematical activity.
3. To learn the mathematics of the students they teach.
4. To learn how to organize possible mathematical environments.
5. To learn the content of variable experiential fields – the mathematical experience of students.
6. To learn the mathematics for the students they teach.
7. To learn how to foster reflection and abstraction in the context of goal-directed mathematical activity.
8. To learn how to encourage students to communicate mathematically among themselves.
9. To learn how to foster student motivation and how to sustain learning over a long period of time.
10. To learn how to communicate pedagogically as well as mathematically with other mathematics educators.

Encouraging reflection and abstraction in learning environments can be very important. I agree with Treffers (1987) that students should be given the opportunity to reflect on their own mathematical experiences by being asked a critical question or by exploration in a novel but related context. In each case, Treffers assumes that at least some successful mathematical learning has occurred so there is something to reflect on – something to be analyzed. Becoming aware of one’s own ways and means of operating when modifying them to meet certain constraints is precisely what is involved in reflection and abstraction and leads to interiorization of the involved operations. In fact, one of my explicit goals when teaching children is for them to become increasingly aware of their own developing mathematical power and autonomy in controlling and monitoring their learning activity. I believe this developing sense of the self doing mathematics successfully and independently is crucial in developing student motivation and in sustaining learning over a period of time. Children need to take responsibility for their own learning as early as possible in their life and they need to distinguish their own conceptual operations and experience from that of others.

We should not confuse interactive communication with mathematical activity. However, the latter can be greatly influenced by the former and this is why I believe that it is crucial for teachers to learn to communicate mathematically with children and how to encourage children to communicate mathematically among themselves (Cobb, 1987; Wood & Yackel, 1990). It is essential that the student’s construction of mathematical reality be a part of their social construction of reality, because it is through the processes involved in socialization that personal mathematical knowledge can be contrasted with the mathematical knowledge of others; and this comparison is the likeliest way to make progress.
XI. FINAL COMMENTS

A goal structure for mathematics education such as the one elaborated by Treffers (1987) is needed in specifying possible learning environments by teachers. But this element of possible learning environments is just as dependent on the experiential fields that constitute learning environments as the latter are dependent on the former. Mathematics educators should not take their goals for mathematics education as fixed ideals that stand uninfluenced by their teaching experiences. Goal structures that are established prior to experience are only starting points and must undergo experiential transformations in actual learning and teaching episodes. Mathematics educators can have a goal structure that initially identifies mathematics for children, yet what that mathematics will eventually consist of is not simply a transformation of the initial goal structure. Hypotheses can be formulated and tested, but mathematics educators should expect to learn certain elements of children’s mathematical knowledge only through their own experiential abstraction. These abstracted goal structures should be modified and made public through interactive communication among mathematics educators for the same reason that mathematical schemes should be modified and made public among children.

As a sociological construct, a possible learning environment is not simply a collection of situations that might be presented to children. It contains a goal structure pertaining to the mathematical knowledge of the students it is intended for, a possible sequence of representative situations the teacher can pose to the students, critical questions that might be asked by the teacher, a possible itinerary of modifications of the involved schemes, and sample interactive communication that should be encouraged among the students. As such, a possible learning environment is a conceptual generalization a teacher can use in the creation of learning environments with his or her students. A learning environment is literally created by its participants and a teacher can use his or her conceptual generalization in this context to guide and interpret children’s mathematical activity. But, as with any other goal-directed activity, the results of the activity can modify the guiding concepts. In other words, a general goal of mathematics teaching is for teachers as well as students to learn, and the primary goal of the constructivist teaching experiment is but a microcosm of this general goal of mathematics teaching.

NOTES

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2. An assimilation is the integration of new objects or new situations and events into previous schemes (Piaget, 1980b, p. 164).
3. A **re-presentation** is the re-creation of a perceptual or motor experience without actual sensory-motor material; one is in it and, in a very real sense, one acts again. It is like a playback.

4. A **scheme** is anticipatory when recognition of an experiential situation leads to a re-presentation of the associated material (or mental) actions and/or result.

5. **Distributive reasoning** can occur in various forms. For example, if a child finds how many candies in five packages with 12 candies in each package by finding the sum of thirty and thirty, that would be an indication of distributive reasoning.

6. In **constructivism**, a zone of potential development of a specific mathematical concept is determined by the modifications of the concept the student might make in, or as a result of, interactive communication in a mathematical learning environment.

7. A **midnight** is an insight that occurs during the activity of solving a problem.

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