

# Children's Fractional Knowledge

Leslie P. Steffe • John Olive

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Leslie P. Steffe  
University of Georgia  
Athens, GA  
USA  
lsteffe@uga.edu

John Olive  
University of Georgia  
Athens, GA  
USA  
jolive@uga.edu

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*We dedicate this book to the memory of Arthur Middleton who was taken from us at the age of 16 by a senseless act of violence. The contributions that Arthur made to our understanding of children's powerful ways of constructing mathematical knowledge will live on in these pages. The world is a poorer place for the lack of what we know he could have accomplished. Our lives and our research are so much richer for having known and worked with him during our teaching experiment.*

# Preface

The basic hypothesis that guides our work is that children's fractional knowing can emerge as accommodations in their natural number knowing. This hypothesis is referred to as *the reorganization hypothesis* because if a new way of knowing is constructed using a previous way of knowing in a novel way, the new way of knowing can be regarded as a reorganization of the previous way of knowing. In contrast to the reorganization hypothesis, there is a widespread and accepted belief that natural number knowing *interferes* with fractional knowing. Within this belief, children are observed using their ways and means of operating with natural numbers while working with fractions and the former are thought to interfere with the latter. Children are also observed dealing with fractions in the same manner as with natural numbers, and it is thought that we must focus on forming a powerful concept of fractions that is resistant to natural number distractions.

In our work, we focus on what we are able to constitute as mathematics of children rather than solely use our own mathematical constructs to interpret and organize our experience of children's mathematics. This is a major distinction and it enables us to not act as if children have already constructed fractional ways of knowing with which natural number knowing interferes. Rather, we focus on the assimilative activity of children and, on that basis, infer the concepts and operations that children use in that activity. Focusing on assimilative activity opens the way for studying reorganizations we might induce in the assimilative concepts and operations and, hence, it opens the way for studying how children might use their natural number concepts and operations in the construction of fractional concepts and operations.

The question concerning whether fractional knowing necessarily emerges independently of natural number knowing is based on the assumption that the operations involved in fractional knowing have their origin in continuous quantity and only minimally involve discrete quantitative operations. In a developmental analysis of the operations that produce discrete quantity and continuous quantity, we show that the operations that produce each type of quantity are quite similar and can be regarded as unifying quantitative operations. The presence of such unifying operations is essential and serves as a basic rationale for the reorganization hypothesis.

We investigated the reorganization hypothesis in a 3-year teaching experiment with children who were third graders at the beginning of the experiment. We selected the

children on the basis of the stages in their construction of their number sequences. Our research hypothesis was that the children would use their number sequences in the construction of their fractional concepts and operations, that the nature and quality of the fractional knowledge the children constructed within the stages would be quite similar, and that the nature and quality of the fractional knowledge the children constructed across the stages would be quite distinct. We did not begin the teaching experiment with foreknowledge of how the children would use their number sequences in their construction of fractional knowledge, nor the nature and quality of the knowledge they might construct. This book provides detailed accounts of how we tested our research hypothesis as well as detailed accounts of the fractional knowledge the children did construct in the context of working with us in the teaching experiment and of how we engendered the children's constructive activity. We do not report on the children who began the teaching experiment in the initial stage of the number sequence because of the serious constraints we experienced when teaching them.

Our overall goal is to establish images of how the mathematics of children might be used in establishing a school mathematics that explicitly includes children's mathematical thinking and learning. Toward that goal, we provide accounts of how the reorganization hypothesis was realized in the constructive activity of the participating children as well as how their number sequences both enabled and constrained their constructive activity. Further, we provide models of children's fractional knowing that we refer to as children's fraction schemes and explain how these fraction schemes are based on partitioning schemes. We found that partitioning is not a singular construct and broke new ground in explaining six partitioning schemes that are inextricably intertwined with children's number sequences and the numerical schemes that follow on from the number sequences. Explaining fraction schemes in terms of the partitioning schemes provides a way of thinking about fractions in terms of children's fraction schemes rather than in terms of the rational numbers. We used our understanding of the rational numbers throughout the teaching experiment as orienting us in our various activities, but we make a distinction between our concepts of rational numbers and our concepts of children's fraction schemes. The former are a part of our first-order mathematical knowledge and the latter are a part of our second-order mathematical knowledge.

Athens, Georgia, USA

Leslie P. Steffe  
John Olive

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This work would not exist, of course, without the collaboration of the school district personnel, school principal, teachers, and students of the anonymous school in which we conducted the Teaching Experiment. We are also grateful to the students' parents and guardians who gave us permission to work with their children over the 3-year period and trusted that we would do our best to help their children move forward in their mathematical thinking.

Finally, we would like to thank our wives, Marilyn Steffe and Debra Brenner, for bearing with us and providing encouragement during our construction of the ideas contained in this book and our writing of it.

Athens, Georgia, USA

Leslie P. Steffe  
John Olive

# Foreword

It is a rare experience in the life of an academic to stand in awe of a body of work. I confess to having had that feeling in the midst of reading Steffe's and Olive's (SO's) account of children's development of fraction knowledge from numerical counting schemes. Their enterprise is especially important, for several reasons – some having to do with fractions and others having to do with science. I'll first say something about the latter and then speak to the former.

## Science

George Johnson (2008) tells the stories of ten experiments that emanated from people questioning accepted wisdom about the physical world and the way it works. His stories are not of individual genius. Rather, the stories are about the scientific method of postulating invisible forces and mechanisms behind observable phenomena, perturbing materials to see if they respond the way your model predicts, and, most importantly, revising your model in light of the specific ways your predictions failed. The stories, above all, are a quest for *understanding*.

Whether research in mathematics education is scientific has been under heavy debate recently. Psychologists, especially experimental psychologists, tend to think it has been unscientific because of its lack of randomized sampling, experimental controls, and statistical analyses. I would argue, however, that it is those who confuse method with inquiry who are too often unscientific. Science is not about *what works*. Science is about *the way things work*. Johnson's stories repeatedly reveal that scientific advances happen when new conceptualizations of phenomena lead to greater coherence among disparate facts and theories. Lavoisier's investigations into the nature of phlogiston, the "stuff" whose release from a substance produces flames, eventually led him to the conclusion that there is no such stuff as phlogiston! After Lavoisier, no one saw combustion as entailing the hidden forces and mechanisms that everyone saw 10 years prior. Were modern psychologists dominant in 1790, they would have criticized Lavoisier for his lack of experimental control. But he had a *strong* experimental control – an initial model of how combustion is supposed to work and of the materials involved



in those processes, and it was his model of how combustion works that he investigated.

It is in this spirit that you must read this book – that SO’s enterprise is to start with, test, and refine their models of children’s fractional thinking. They also take seriously the constraints of employing a constructivist framework for their models, predictions, and explanations: Children’s mathematical knowledge does not appear from nothing. It comes from what children know in interaction with situations that they construe as being somehow problematic. To be scientific in this investigation, SO take great pains to give precise model-based accounts of the ways of thinking that children bring to the settings that SO design for them.

Which brings up another point. The significance of children’s behaviors can only be judged in the context of the tasks with which they engage and as they construe them. In fact, how a child construes a task often gives insight into the ways of thinking the child has. I urge readers to read SO’s tasks carefully and to understand the computing environment that gave context to them. The computing environment (TIMA) afforded actions to children that are not possible with physical sticks, and hence children were able to express anticipations of acting that would not have been possible outside that environment. Read the tasks slowly so as to imagine what cognitive issues might be at play in responding to them. I urge you also to read teaching-experiment excerpts slowly. SO’s models afford very precise predictions of children’s behavior and very precise explanations of their thinking, so the smallest nuance in a child’s behavior can have profound implications for the theoretical discussion of that behavior. For example, according to SO’s models of number sequences, if a child partitions a segment into 10 equal-sized parts, but has to physically iterate one part to see how long 10 of them will be, this has tremendous implications for the fraction knowledge we can attribute to him or her. The contribution of SO’s work is that their theoretical framework not only supports such nuanced distinctions, but also allows us to understand what might appear to some as uneven fraction knowledge instead as a coherent system of thinking that has evolved to a particular state (and will evolve further to states of greater coherence).

Finally, it is imperative in reading this book that you understand that *SO employ teaching as an experimental method*. Understanding this, however, requires an expanded meaning for experiment and a nonstandard meaning for teaching. The idea of a scientific experiment is to poke nature to see how it responds. That is, we start with an idea of how nature works in some area of interest, perturb nature to see whether it responds in the way our understandings would suggest, nature responds according to its own structures, and then we revise our understandings accordingly. It is in this respect – teaching as a designed provocation – that it can be used as an experimental method in understanding children’s thinking. To be used effectively as an experimental method, though, you cannot think of teaching as a means for transmitting information to children. Rather, you must think of it as an interaction with children that is guided by your models of children’s thinking and by what you discern of their thinking by listening closely to what they say and do. Of course, all this is with the backdrop that children are participating according to their ways of thinking and with the intent of understanding your, the teacher’s, actions.

## Fractions

SO's basic thesis is that children's fraction knowledge can emerge by way of a reorganization of their numerical counting schemes. This might, at first blush, seem like a weak hypothesis, as in "you can devise super special methods and invest super human effort to have students create fractions from their counting schemes." I propose a different interpretation:

*If allowed*, children can, and in most cases will, use their counting schemes to create ways of understanding numerical and quantitative relationships that we recognize as powerful fractional reasoning.

The phrase "if allowed" is highly loaded. It does not mean that children should be turned loose, with no adult intervention, to create their own mathematics. We know that little of consequence will result. Rather, it means that the instructional and material environments must be shaped so that they are amenable to children using natural ways of reasoning to create more powerful ways of reasoning – they are designed to respect children's thinking and build from it.

There are three important aspects to SO's argument for the reorganization hypothesis. The first is that they did not start with it. Rather, it emerged from their interactions with children. In a sense, the children forced the reorganization hypothesis upon SO. Children whose number sequences did not progress to higher levels of organization simply were unable to progress in their fraction knowledge despite SO's best attempts to move them along. Children whose number sequences were limited, developmentally speaking, to early forms simply could not see fraction tasks in the ways that children with the generalized number sequence could.

Second, the reorganization hypothesis entails the claim that children's number sequences are very much at play as they develop spatial operations with continuous quantities. It is through their number sequences that children impose segmentations on continuous quantities and reassemble them as measured quantities.

Third, SO's reorganization hypothesis removes any need to think that the operation of splitting, as described by Confrey, appears independently of counting. In a very real sense, SO's explication of the reorganization hypothesis gives Confrey's work a developmental foundation. But it does more. As noted by Norton and Hackenberg (Chap. 11), the splitting operation described by Confrey is not sufficient for children to generate the highest level of fraction reasoning described by SO. More is required, and SO give a compelling argument for what that is.

## Next Steps

Norton and Hackenberg (Chap. 11) give a highly useful analysis of potential connections between SO's research on fractions with other research programs in the development of algebraic and quantitative reasoning. My hope is that SO's research develops another set of connections – with pedagogy and curriculum. What sense

might teachers make of the reorganization hypothesis? What reorganizations must they make to understand it and to use it? What professional development structures could help them understand and use it? How could the reorganization hypothesis inform the development of curriculum that in turn would support teachers as they attempt to actualize the reorganization hypothesis? I look forward to SO and protégés giving us insight into these questions.

Tempe, Arizona, USA

Patrick W. Thompson

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