In Focus…
What is Required to Understand Fractal Dimension?

Patrick W. Thompson

The idea of fractal dimension is based on the idea of Euclidean dimension. But understanding this connection is harder than you might think. Children often think of areas and volumes in a way that we would describe as "one-dimensional objects" (see Figures 1 and 2). What is, for example, one-dimensional area? It is a conception of a measurement of a region as being "how many squares do you need to lay down to fill a region," where the squares are, in the child's conception, no more than, say, pieces of paper with sharp corners. That is, they do not conceive of the unit as a dimensioned object. It is simply an object.

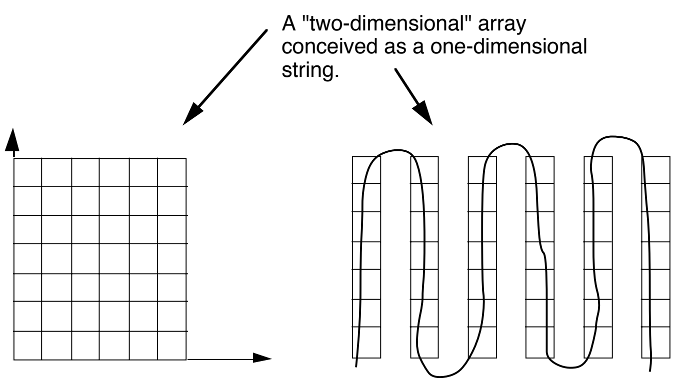

Figure 1. One-dimensional area.

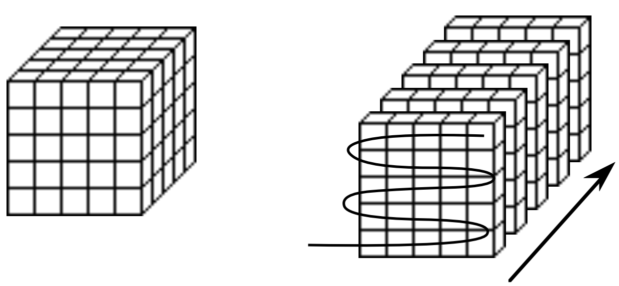

Figure 2. One-dimensional volume.

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For example, in one study, I asked 6 fifth-graders to consider the area of the rectangle in Figure 3. They all said that they must either convert centimeters to inches or inches to centimeters before doing anything else. I asked the question, "Suppose we did the silly thing and just multiplied 4 and 3. We would get 12. But 12 what?"

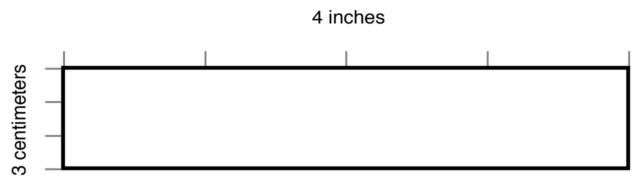

Figure 3. Multiply 4 by 3. You get 12. Twelve what?

The ensuing discussion went on for 35 minutes before one child asked timidly, "Would it be 12 rectangles that are 1 cm by 1 inch?" In the next 10 minutes children worked to understand how it was that (1) it made sense, in multiplying length by width, they were somehow generating rectangles, and (2) that the only thing they needed to know about a "covering collection" was the unit-length of each side of the basic area unit.

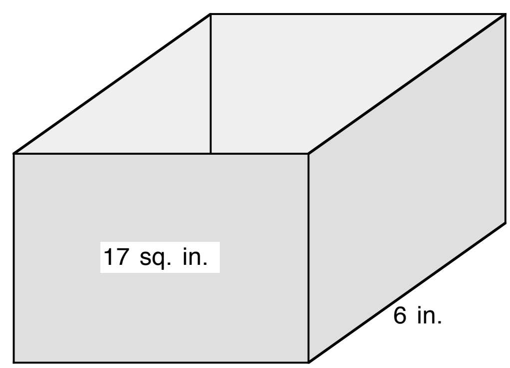

Figure 4. What is the volume of the box?

In another set of interviews, which followed an intensive teaching unit which took a standard approach to developing formulas for area and volume (i.e., counting squares in rows and columns, or counting cubes in rows, columns, and layers), only one student could answer the question associated with Figure 4. He said, "Oh, somebody has already done the multiplication for me!" and went on to multiply 6 by 17. Upon further probing, he said that he didn't need to know the other side lengths because "if I had them I'd just multiply, and when I multiply I would just get 17, and I've already got 17." Other children responded that they needed to know the other side lengths. I asked them what they would do if they knew the side lengths. They said that they would multiply the side lengths, and they couldn't know what they would get unless they knew the actual numbers.

These examples illustrate my claim that children often have not made a conceptual connection between the formulas they use for area and volume and actual area and volume. It also illustrates my claim that children need to conceptualize their units of measurement as dimensionalized objects in order to understanding the ideas of area and volume as "dimensionalized attributes" of objects (see Figure 5). Otherwise, typical encounters with activities to develop area and volume formulas end up capturing nothing more than "quick counting" techniques to determine the number of objects they would otherwise count one-by-one (see Figures 1 and 2).

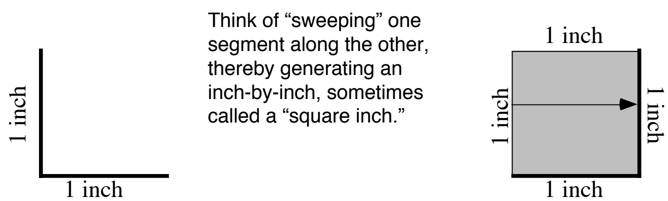

Figure 5. A "square inch" generated by two segments each 1 inch long.

Now, suppose that students have developed "dimensionalized attributes" conceptions of area and volume. Then it makes sense to extend their notion of dimension from "independent directions of sweep" to dimensionality as an invariant relationship between replication and similarity.

The idea of replication is to build a similar copy of the figure you have out of identical copies (Figure 6). The idea of similarity is that there is a multiplicative expansion (e.g., a "blow-up" of photograph) so that each linear component is k times as long in the blow-up as in the original. If r is the number of copies used to replicate a figure, and if k is the scale factor (each linear component of the replicate will be k times as long as its original), then, in Euclidean space $r = k^d$ for some whole number d. It is absolutely essential that if students are going to understand the idea of fractal dimension as an extension of Euclidean dimension, then they must internalize this relationship as one *they insist must hold for all similar figures*. That is, they must come to think of it not as just a generalization, but as a defining relationship between similarity and dimension.

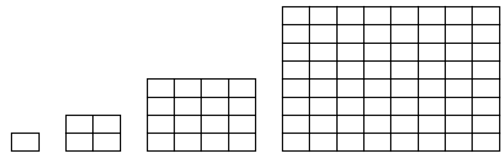

Figure 6a. Each successive replicate is made with 4 copies of the previous figure. The scale factor between successive replicates is 2.

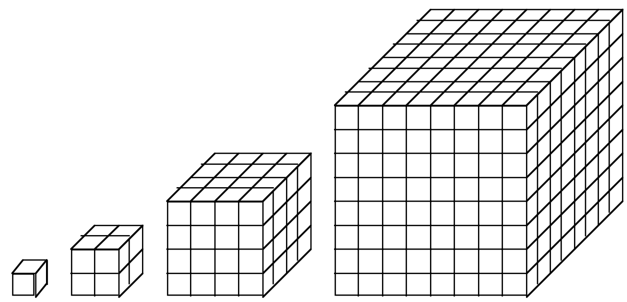

Figure 6b. Each successive replicate is made with 8 copies of the previous figure. The scale factor between successive figures is 2.

Another understanding students will need to have in order to understand fractal dimension is the idea of self-similarity. It is a non-trivial leap to believe that the two figures in Figure 7 are similar *in every detail*. For example, students will often think that, since the larger figure is made from five copies of the smaller figure, *the larger one has five times as many pieces in it, so the two cannot be similar*. To accept that they are identical in detail, students must have a mental image of how fractals can be generated, with that image entailing the fractal itself as being the *limit* of the generating process.

To continue, you need 5 copies of the smaller curve to make the larger replicate, and the scale factor is 3 (the larger curve lies on a segment 3 times as long as the segment on which the smaller one lies). If we believe that the curves in Figure 7 are similar, then we can generalize from the Euclidean case to conclude that $5 = 3^d$, where d is the curve's dimension—but not necessarily a whole number. The number d which gives $5 = 3^d$ is $d = \frac{\log 5}{\log 3}$, or $d = 1.46$.

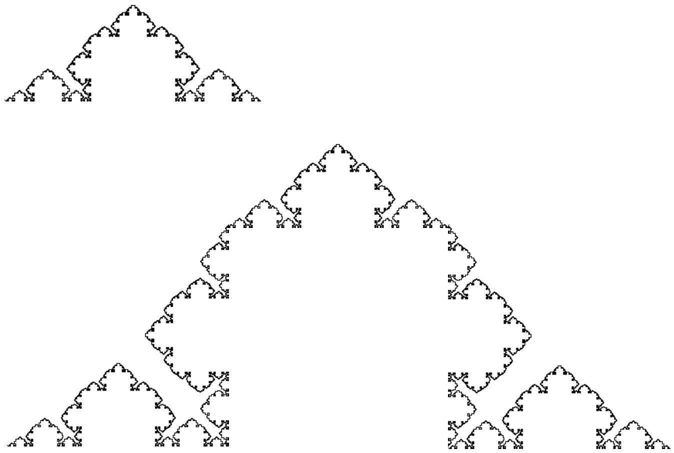

Figure 7. Are these two curves similar *in every detail*? In particular, does the larger curve have 5 times as many pices in it as the smaller curve?

In summary, I claim that to understand fractal dimension as a generalization of Euclidean dimension, students must have interiorized the relationship between scale and replication stated below to the point it is so obvious that it cannot be questioned.

If r is the number of copies used to replicate a figure, and if k is the scale factor (each linear component of the replicate will be k times as long as its original), then, in Euclidean space, $r = k^d$ for some whole number d.

That students reach the point that this relationship is obvious is highly non-trivial. I suspect that interiorizing this relationship is tantamount to developing a full scheme for multiplictive reasoning. Sounds like a promising dissertation topic.

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Conferences…

PME-NA XXII, Twenty-second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education; October 7-10, 2000 in Tucson, Arizona; http://www.west.asu.edu/cmw/pme/

ICTCM, 13th International Conference on the use of Technology in Collegiate Mathematics; November 16-19, 2000 in Atlanta, Georgia; http://www.ictcm.org

TIME 2000-An International Conference on Technology In Mathematics Education; December 11-14, 2000 in Auckland, New Zealand; http://www.math.auckland.ac.nz/Conferences/TIME2000/firstann.html

CERME 2, Second Conference of the European Society for Research in Mathematics Education; February 24-27, 2001 in Mariánské Lázne, Czech Republic; http://www.erme.uni-osnabrueck.de/cerme2.html

79th Annual Meeting of NCTM; April 2-7, 2001 in Orlando, Florida; http://www.nctm.org/meetings/annuals/orlando

PME 25. Twenty-fifth Annual Meeting of the International Group for the Psychology of Mathematics Education; July 12-17, 2001 in the Netherlands; http://www.fi.uu.nl/pme25/