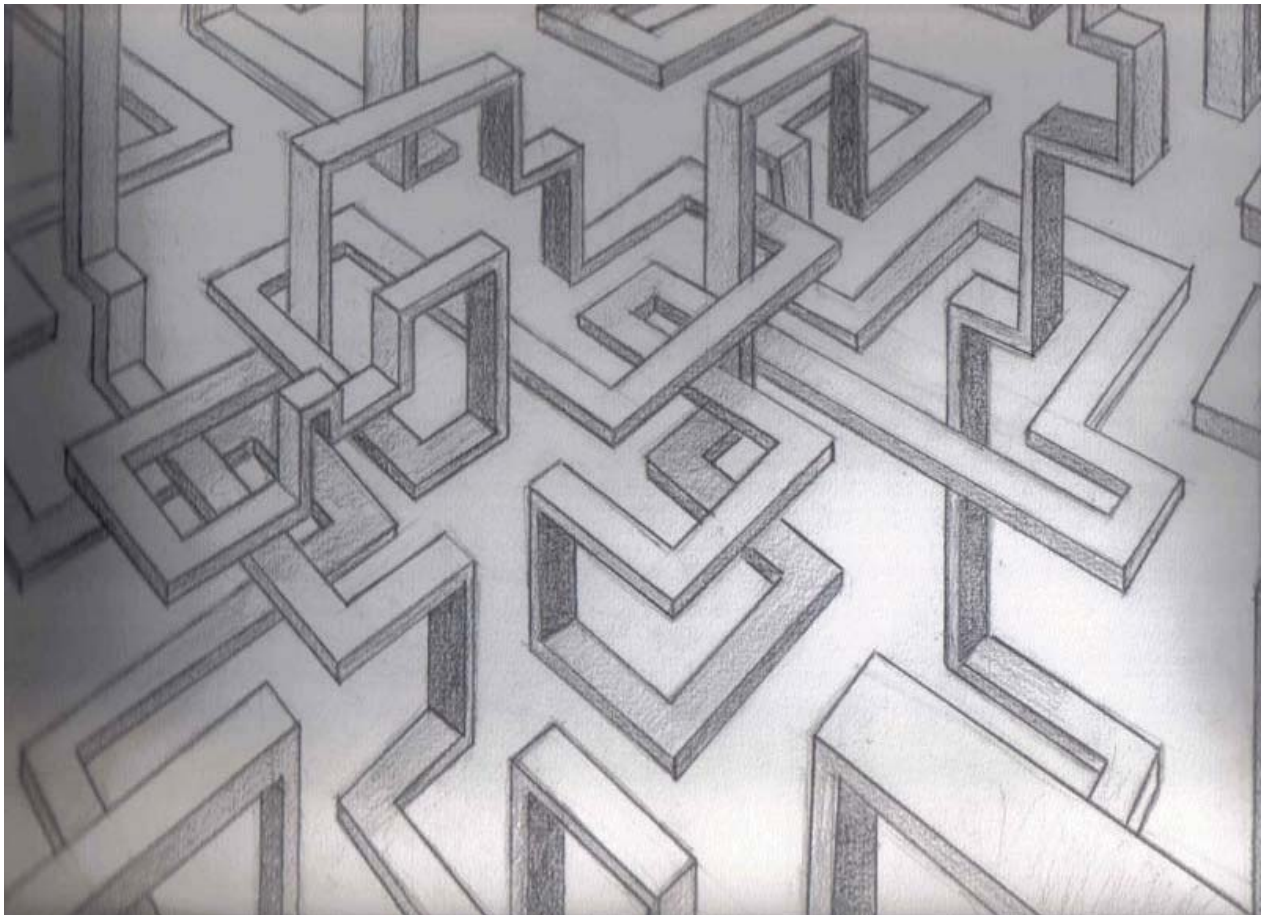


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## A Note from the Editors

Dear *TME* Readers,

On behalf of the editorial staff and the Mathematics Education Student Association at The University of Georgia, we are pleased to present you with the first issue of the 20th volume of *The Mathematics Educator*. We hope you enjoy the mixture of articles in this opening issue. Our authors have used both quantitative and qualitative methods to investigate links between mathematics education and college readiness, reading ability, the way research mathematicians think, and the study of martial arts. Once again, we are reminded of the myriad factors affecting and affected by mathematics education.

We open this issue with a guest editorial by Rachael Eriksen Brown in which she explores the competing tensions of leading a professional development program. She poses several questions to the mathematics education community for further consideration. The difficulties she describes will be familiar to both classroom mathematics teachers and university faculty. Our first article, by Jeremy Zelkowski, examines the impact of secondary mathematics scheduling options on college-readiness. Drawing from his experience as a college mathematics professor, he posits that continuous enrollment in secondary mathematics will better prepare students for college mathematics. Our readers will appreciate that his distinctions between the various implementations of continuous enrollment and his thoughtful consideration of what it means to be college-ready. The second article, by John H. Lamb, analyzes the effect of reading difficulty on student performance on mathematics assessment items. He uses analysis of covariance on results from the Texas Assessment of Knowledge and Skills (TAKS) to form his conclusions. In contrast to these two articles focused on specific issues of interest in mathematics education, Serkan Hekimoglu offers a broader perspective on the study of mathematics. In his article, he compares the study of mathematics to the practice of martial arts and details how his understanding of martial arts has influenced his teaching practice. Our final article by Revathy Parameswaran is a qualitative study of the cognitive tools expert mathematicians use to understand abstract definitions. Dr. Parameswaran also explains the pedagogical implications of these findings. We close this issue with Eileen Murray's review of *Mathematics Education at Highly Effective Schools That Serve the Poor*. She calls it a "powerful example" of how schools can meet the needs of traditionally marginalized populations.

We would like to thank our associate editors and authors for all their hard work and dedication. And we would like to thank our reviewers for their helpful feedback on manuscripts. Without these tireless volunteers, our work would be impossible. We hope you enjoy reading this issue as much as we all have enjoyed working on it.

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## About the cover

The artwork on the front cover is a drawing provided by Kylie Wagner, who specializes in artwork using mathematical ideas. This piece relies on geometric imagery to produce an optical illusion.

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## Guest Editorial...

# Tensions Faced by Mathematics Professional Developers

Rachael Eriksen Brown

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I recently worked on a research project in which I was the facilitator of a middle school mathematics professional development course, InterMath Rational Numbers, and part of a research team examining InterMath through a National Science Foundation-funded project, Does It Work?: Building Methods for Understanding Effects of Professional Development (DiW). In my position as one of the InterMath facilitators for the DiW project, I took a leading role in tailoring the original InterMath syllabus to meet the needs of the DiW project and the participating teachers. While the course was a great experience for me, it also brought up several tensions that, I suspect, often exist in mathematics education at all levels. In particular, there were several goals in the course that did not appear to be in opposition, yet led to many tensions in my actual practice. After describing my experience I will pose several questions to the mathematics education community about the interwoven, yet sometimes conflicting, goals we often have for our classes.

Four common recommendations for effective professional development (PD) include a focus on mathematical content, the use of activities that actively engage teachers in learning, planning for sustained time to learn, and developing a community of learners (e.g., Garett et al., 2001; Guskey, 2003; Sowder, 2007). However, little has been written about the tensions that arise for the mathematics professional developer who is attempting to balance content coverage with elements of effective PD. I felt that the InterMath (IM) syllabus gave me the time and opportunity to balance all four PD recommendations. Hence, my goal was to be as true to the modified syllabus as possible. Nonetheless, as the weeks passed, the pressure to examine all of the content topics started to come into conflict with my desire to balance the other elements of effective PD. I wanted the course to be successful not only for the participants, but also for the DiW research project. This meant I wanted to ensure quality content

coverage for the DiW research project, make the course engaging, and support the development of a community of learners for the teachers. In this commentary, I use literature and personal experience to describe the tensions between the different PD goals and pose questions to the mathematics education community for consideration about PD aimed at building teachers' mathematical content knowledge.

### Effective Professional Development

Many studies have shown the need to meet with teachers multiple times to have the greatest impact on teacher learning and change in teaching practice. Garett et al. (2001) found time span and contact hours were important features in PD because both of these measures had a positive influence on opportunities for active learning and focus on content knowledge. Garett et al. found "professional development is likely to be of higher quality if it is both sustained over time and involves a substantial number of hours" (p. 933). Additionally, Banilower, Heck, and Weiss (2007) found the effects of PD to be the greatest when contact hours were high. Their results suggested that if contact time was between 32 and 80 hours teachers would gain the most from the PD. Besides contact hours, this recommendation implies activities designed for teachers should be high quality (focusing on active engagement with content knowledge). Guskey (2003) noted, "although effective professional development clearly requires time, it also seems clear that such time must be well organized, carefully structured, and purposefully directed" (p. 12). Thus, the challenge for professional developers is to ensure that sufficient time is being spent on each mathematical concept. While the IM course included 40 contact hours, well within the recommended 32 to 80 hours, I still struggled with how to distribute the time allotted to various mathematical content.

An important guideline for PD is to focus on specific content (Guskey, 2003; Sowder, 2007). Garett et al. (2001) reported that PD focusing on content knowledge "is more likely to produce enhanced knowledge and skills" (p. 935). Research has also supported the notion that focusing on improving

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teachers' content knowledge in PD has the potential to positively impact student learning (Hill, Rowan, & Ball, 2005). Cohen and Hill (2000) found empirical evidence to support content knowledge as the focus of PD in changing teachers' practices, noting, "It seems to help to change mathematics teaching practices if teachers have even more concrete, topic-specific learning opportunities" (p. 312).

Rational numbers is an important topic in middle grades mathematics in the United States, and research has shown teachers' understandings of these concepts are generally not strong (e.g., Ma, 1999; Post, Harel, Behr, & Lesh, 1998). Because of this, the DiW team focused the IM course on this topic. The assumption was that an increase in teachers' mathematical knowledge for teaching would translate into a positive impact on their teaching practices and their students' understanding of mathematics. The team decided on three themes for IM: referent unit, drawn representations, and proportionality.<sup>1</sup> Thus, consistent with recommendations for effective PD, IM explicitly focused on mathematical content.

For the professional developer, focusing on content means planning activities where mathematics is the focus, and supporting and encouraging teachers as they engage in the mathematics that they teach. For example, Kazemi and Franke (2004) studied a group of teachers who used students' written work as a springboard for discussing their students' responses to a problem provided by the facilitator. The facilitator's role was to press teachers to focus on student strategies and propose strategies the teachers did not suggest. They observed that the facilitator was able to help guide the direction of the meetings and assist the group in maintaining a focus on students' mathematical work. My goal was to press the teachers to remain focused on mathematics and actively engaged in the selected mathematical tasks while also keeping them engaged on working to develop community.

The need for PD to help activate knowledge in the participants, not to deliver knowledge, was a common theme in Wilson and Berne's (1999) review of PD literature. Active learning in PD includes working together, sharing ideas and strategies, and becoming reflective practitioners. In IM, teachers worked in pairs

and as a whole group on cognitively demanding tasks focused on rational numbers. By using these tasks, we were addressing many of the PD recommendations: having the teachers engage with mathematically demanding problems, work collaboratively, and do so over a sustained period of time. Professional developers "engage them [teachers] as learners in the area that their students will learn in but at a level that is more suitable to their own learning" (Wilson & Berne, 1999, p. 194). Thus the professional developer has to support teachers in acknowledging their lack of understanding on material they are responsible for teaching, while motivating them to engage with the mathematics they are to learn. Without teachers being willing to engage as learners, IM would have been a failure because the design of the course was dependent on the teachers working together on tasks with little to no direct instruction from me.

Borko and Putnam (1995) noted successful PD provides "opportunities for teachers to construct knowledge of subject matter and pedagogy in an environment that supports and encourages risk taking and reflection" (p. 59). This characteristic includes developing an intellectual space where teachers can make public their understandings, as well as misunderstandings. Community building activities are common experiences in PD that supports meaningful interactions between participants and the facilitator, and building community is an important component of effective PD. In Kazemi and Franke's (2004) study, the teachers began to develop norms about what it means to teach and learn from each other based on their examination of student work. This development of norms was related to the community building of the group and their active engagement in examining students' mathematical work. Active learning lends itself to creating a community of practice as teachers are given opportunities to explain, compare, and contrast mathematical strategies for solving tasks as a group.

Because of my passion for community development, this was important to me as I planned IM. The design of the modified syllabus for the IM Rational Numbers course was especially important because the pacing and tasks allowed me time and opportunity to build a community. For example, there was time at the beginning of the course for me to lead a discussion about the expectations participants had about IM, myself as the facilitator, and themselves as participants. In addition, I felt passionate about all the tasks and the sequencing of the concepts. When learners are producing high-quality materials and are

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<sup>1</sup> *Referent unit referred to the whole for a given quantity. Drawn representations included array models, area models, single number lines, double number lines, tables, and graphs. These representations were intended to be used to reason about a given problem, not just as a picture of the solution. Proportionality referred to multiplicative reasoning in fraction and decimal operations as well as situations involving direct and inverse proportions.*

engaged in mathematical discussion, the facilitator is, at least implicitly, working to build a community. I felt confident that the syllabus would allow for both adequate time on mathematically-focused activities and community building, hence balancing all four recommendations.

### Tensions

As the facilitator I felt a constant tension between my intentions of increasing participant's mathematical content knowledge (the explicit goal of IM) and building a mathematical community of practice (my personal goal as facilitator). Balancing community building and pushing the participants' mathematical knowledge was a challenge. I also felt tremendous responsibility to keep the teachers satisfied with the experience in order to continue to motivate them to engage deeply with the mathematics together. For me, part of the satisfaction would be to continually relate what they were experiencing to their own classroom practices. This meant being explicit about how my actions could be replicated in a classroom of middle grades students. Finally, the pacing of the PD and my knowledge of teachers' understanding provided a tricky dilemma for me to navigate.

One of the principal ways I tried to build community was by fostering productive conversations as we debriefed our mathematical work on tasks in the whole group setting, but I rarely had enough time for all the tasks and conversations I had planned. Thus, I was faced with making decisions about doing more tasks to meet the goal of developing content knowledge versus doing fewer tasks and having longer conversations about them to meet my goal of building community. The tension between building the teachers' mathematical knowledge and building community was overwhelming at times and made me feel like I was unable to be successful on either front. In one of my journal entries I wrote, "There is so much that we needed to talk about and do but we didn't have time. I'm concerned because we move on to division of fractions next. I feel like I'm leaving many participants with holes in their knowledge."

Because of the time crunch and the tug I felt between debriefing the mathematics and debriefing our conversations, I never explicitly addressed my community building efforts with the participants. Although I praised the group at the conclusion of a good conversation, we never discussed what made those conversations special in terms of either building a community or developing mathematical knowledge. Despite my efforts to balance building content knowledge with building community, there were times

when one goal dominated my decision-making in our class meetings.

This leads to another tension I faced about how explicit to be about what I was asking them to do as the facilitator and the success we were having in learning about rational numbers. Clearly, I struggled to do this related to community. But the time limits also impacted how often we could discuss how what they were experiencing could be implemented in their classrooms. This was important to me because I saw it as a component of active learning. For example, the National Council of Teachers of Mathematics (NCTM) has been promoting a vision of the ideal mathematics classroom as one where teachers "establish and nurture an environment conducive to learning mathematics through the decisions they make, the conversations they orchestrate, and the physical setting they create" (NCTM, 2000, p. 18). One goal of mine was to provide an example of learning in this environment. How explicit should I be about how I did this? In the end, there was little time during class to have these conversations. By not finding the time to discuss our own practices as a community, the teachers were unable to reflect as a group on the behaviors and actions that supported our development of a nurturing, mathematical environment. When working with the teachers in small groups, I would ask about their classrooms and students and suggest that what I was doing could be done in their classrooms. Reflection components were part of the course; teachers were asked to do a ticket out the door after each class meeting and were interviewed by a researcher each week about the course. Through these reflections, teachers did think about my role as the facilitator. For example, one teacher said, "She has the approach that you know the answers, there are no right or wrong answers. But everyone should be able to learn from each other. And she tries to make sure that she is not the center focal point, is the center focal point, that it is on us, the students" (King, Week 8). The teachers, however, rarely made connections between what they were experiencing and their own classrooms in the phone interviews or in our class discussions.

IM engaged teachers in exploring high cognitive demand tasks related to rational numbers, and at times these tasks pushed the participants to their mathematical limits, which was often uncomfortable. This discomfort was not surprising, as research has shown that exploring rational number concepts is often an uncomfortable enterprise (Armstrong & Bezuk, 1995; Ma, 1999). The discomfort had the potential to inhibit our work together because participants might

have been reluctant to expose the gaps in their knowledge to their peers, particularly because those peers taught in the same district. The participants, however, worked through this discomfort and grew to expect it when working together in IM. For example, one task the teachers were asked to work on had a lower cognitive demand. The participants noticed the change in the nature of the task because they expressed concern that the task felt too easy, and they were sure they had missed something. The tension I then faced was determining whether to continue to follow the syllabus despite my growing awareness of teachers' misconceptions about rational numbers and their willingness to continue to engage deeply with the mathematics. This tension speaks to the recommendation for sustained time to learn. If I forged ahead, how would that affect the quality of our mathematical work? If I remained focused on the same content, would the teachers continue to be actively engaged in the material? This tension was complicated by the fact that the PD was being offered as part of a research project and the district had been promised a specific course. Thus, as a larger team, the decision was made to continue following the pace of our syllabus.

### Concluding Remarks

Despite the tensions I felt as the facilitator, IM was a success. Not only did the teachers report being happy with the experience, but the increase in the average score on the pre-assessment and the post-assessment of their mathematical knowledge for teaching rational numbers was statistically significant. More specifically, nine participants had a significant increase in their scores. In addition to the quantitative data, there was qualitative evidence that teachers grew in their understanding of the three themes of IM—referent unit, drawn representations, and proportion—and that the class was able to form a community of practice. Further analysis is being conducted on other aspects of teacher learning in IM.

I was able to learn more about myself as an educator and about the challenges faced by facilitators as they attempt to focus on increasing mathematical knowledge and community building. The tensions faced by the professional developer who is balancing the development of mathematical knowledge and following guidelines for effective PD are not trivial. I would like to challenge the mathematics education research community to seriously consider the potential tensions faced by the mathematics educator who finds herself managing these dual, sometimes opposing, goals. I pose the following questions for consideration:

1. How does this tension influence the professional development experience?
2. What can we learn about teaching, in general, from our experiences as facilitators?
3. What can we learn about community building?
4. What does the facilitator learn through this experience? How does it impact how she conducts the PD?
5. How can these understandings be translated to classroom teachers' experiences?

As our understanding of PD grows (including learning of teachers, fidelity of programs, and essential features for effectiveness), we need to consider how the simple decision to facilitate the PD being examined influences all who are involved in the experience.

### Acknowledgments

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# Secondary Mathematics: Four Credits, Block Schedules, Continuous Enrollment? What Maximizes College Readiness?

Jeremy Zelkowski

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This paper posits the position that if higher education and secondary schools wish to increase students' college readiness, specifically in mathematics and critical thinking skills, continuous enrollment in secondary mathematics is one avenue worth exploring as opposed to increasing mathematics graduation requirements only in terms of Carnegie credits. NAEP-HSTS 2005 and NELS:88 data indicate, respectively, non-continuous enrollment in secondary mathematics results in lower mathematics achievement and decreases the odds of completing a bachelor's degree. Nationally, schools following 4×4 block schedules (90-minute classes that meet daily for only one semester) were found to have mathematics achievement scores two thirds of one grade-level lower than schools following a 50-minute year-long mathematics courses. Typical college-bound students who do not take mathematics all four years of high school likely diminish their odds of bachelor degree completion by about 20%.

State and district education policies concerning the number of high school mathematics credits required for graduation vary widely (Education Commission of the States [ECS], 2007). Further, higher education institutional admissions criteria also vary broadly between institutions. Over the course of the last 10 years, many states have increased the number of mathematics credits required for high school graduation from the three credits recommended by the *Nation at Risk* report (National Commission on Excellence in Education, 1983) to the now popular policy of four mathematics credits. However, among states with the increased requirement, only a few have a written policy that requires college bound students to be enrolled in a mathematics class each year of high school. The policy language tends to focus more on accumulation of credits than mathematics achievement. Moreover, I argue the position that block scheduling in high schools (see “Secondary School Class Scheduling Formats”) may well be contributing to the stagnant college graduation rates and remediation issues well-documented over the past decade (e.g. Adelman, 2010; Attewell, Lavin, Domina, & Levey, 2006; Horn, 2006; Hoyt & Sorensen, 2001; Knapp, Kelly-Reid, & Ginder, 2010).

Adelman (1999, 2006) indicated that secondary mathematics is the predominant predictor of bachelor degree completion using two national longitudinal data sets—High School and Beyond (HS&B) and the

National Education Longitudinal Study (NELS). This research extends Adelman's contention by studying secondary course enrollments in mathematics to test the conjecture that continuous enrollment in high school mathematics is linked to bachelor degree completion. If this conjecture were to hold up, then institutes of higher education, state departments of education, and school districts should consider implementing a continuous enrollment policy for mathematics. Moreover, schools and districts may reexamine whether the popular 4×4<sup>1</sup> block schedule format, present in nearly one-third of secondary schools (National Assessment of Education Progress [NAEP], 2009), is actually increasing students' mathematics achievement or decreasing it and thereby possibly reducing college readiness.

## Conceptual Framework

### *College Readiness*

For nearly 75 years, researchers have been studying *critical thinking* (Browne, Haas, Vogt, & West, 1977; Ennis, 1993; Facione & Facione, 1994; Glaser, 1941; Jones & Ratcliff, 1993; Norris, 1989, 1990, 1992; Williams & Worth, 2001). Glaser (1941, pp. 5-6) defined critical thinking to include: (a) an attitude of being disposed to consider in a thoughtful way the problems and subjects that come within the range of one's experiences; (b) knowledge of the methods of logical inquiry and reasoning; and (c) some skill in applying those methods. More recently, researchers (Ennis, 1993; Facione & Facione, 1994; Williams & Worth, 2001) have tried to narrow the definition and measure a person's ability to think critically. Though a universal definition of critical

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thinking currently does not exist, many college and university faculty agree that the ability to think critically is central to success in college regardless of course content. Thompson and Joshua-Shearer (2002) surveyed and interviewed college students and reported two overwhelming conclusions: high schools need to teach better critical thinking and study skills. In the same study, the teachers most frequently labeled by students as *their worst teachers* in high school were mathematics and science teachers. In light of Thompson and Joshua-Shearer's findings, it might be expected that average college-bound students looking to be *college-eligible* rather than *college-ready* would avoid challenging mathematics and science courses in high school if policy permitted such action.

David Conley (2005) reported the distinctive environment of U.S. high schools focuses more on getting students *college-eligible* rather than *college-ready*. College-eligible refers to meeting a state's minimum high school graduation requirements *and* public college admission requirements. College-ready, on the other hand, refers to meeting a state's highly recommended course-taking suggestions to improve college-readiness, completing rigorous advanced core subject courses during the senior year of high school, *and/or* meeting the minimum college entrance test scores predicting successful completion of entry-level college core courses.

Conley (2005), arguably the leading U.S. researcher in this field, has led a charge in trying to build a bridge from secondary school coursework to college coursework. His Standards-For-Success (S4S) project is the most extensive compilation of specific *college-readiness* standards for the core high school subjects. However, many students today may still avoid challenging courses late in high school knowing they have reached *college-eligibility*. ACT<sup>ii</sup>, Inc. (2005) indicates only 22% of students who complete the three basic core mathematics classes of Algebra 1, Geometry, and Algebra 2 (or the equivalent) will meet the ACT *college-readiness* benchmark (22 out of 36 on the ACT mathematics component), which, ACT indicates, is the tipping point predictor score of successful completion of an entry-level college algebra course. In secondary schools requiring a fourth credit for graduation, students can easily fulfill the final mathematics credit with half-credit mathematics electives (special topics courses) that may fail to increase *college-readiness*. "Typically, students (even the brightest ones) avoid tasks that appear to require more energy than the students are willing to expend" (Sparapani, 1998, p.1). Hence, although increasing

graduation credit requirements in mathematics might seem to be a fix to the situation, students can still avoid challenging mathematics in preparation for college. Schools following the 90-minute 4×4 block schedule format present even more non-continual enrollment concerns.

## Secondary School Class Scheduling Formats

### *Traditional Period Scheduling*

Traditional period scheduling in high schools typically follows a schedule where students attend one-credit classes for the entire school year. Typically, each class meets for 50 to 55 minutes daily, for a total of seven class periods per day. In some high schools that are forced to have several lunch periods, longer classes are possible. In this case, the high schools may have an eight period day where teachers have two planning periods and teach six class periods. In an eight period day, students typically have one period as a study hall or activity period, but they could also have academic classes for all eight periods. There are other variations as well.

### *Block Scheduling*

Block scheduling can follow different scheduling formats as well. I will briefly explain three of them here: The most common is the 4×4 *block* spanning 80- to 90-minute class periods. Students take four full credit courses daily each semester of the year. Some school systems divide their year into trimesters. Students essentially take a full credit course over the span of two consecutive terms in the trimester. Period lengths vary from 70- to 90-minutes for trimester formats.

In the *alternating block*, or the *A/B block*, students alternate between classes every other day, similar to a college format. However, classes meet all year instead of during just one semester. One week, students attend A block classes on Monday-Wednesday-Friday and B block classes on Tuesday-Thursday. The following week, the A and B blocks switch days of the week. The typical class length can range depending on the day of the week, but generally falls into the 70-90 minute range. There are other variations on how the days alternate.

Finally, the *Copernican block* schedule, a format, used in a very small percentage of high schools, has two configurations. In the first configuration, a long four-hour block consists of a main core course like English or Algebra. Then, two or three shorter 1 to 1.5 hour classes fill the rest of the day. Every month, students switch schedules where courses change to

balance out the curriculum time for each of the four major content areas. The second configuration takes the four-hour block into two two-hour blocks. The rest of the day is the same as the first configuration. In this configuration, students switch schedules every two months instead of monthly.

Arguably, block scheduling may be the single greatest sign of reform teaching strategies implemented since the early 90s in American high schools because the format theoretically offers longer class time to go deeper in the content during the extended 30-plus minutes of class time over traditional 45- to 55- minute class periods. However, the research community is confused by the mixed findings of block scheduling. Some research (Buckman, King, & Ryan, 1995; Duel, 1999; Fletcher, 1997; Khazzaka, 1998; Queen, Algozzine, & Eaddy, 1997; Snyder, 1997; Stader, 2001) has shown support for block scheduling formats, claiming improved achievement, student attendance, and school climate and a decreased incidence of discipline problems. Equally persistent has been research asserting block schedules produce negative results or fail to demonstrate any statistically significant improvements over other scheduling formats (Cobb, Abate, & Baker, 1999; Fletcher, 1997; Hamdy & Urich, 1998; Knight, De Leon, & Smith, 1999; Lare, Jablonski, & Salvaterra, 2002; Pisapia & Westfall, 1997). The research does agree that the subject being taught, the teacher's ability and knowledge, and school climate are *all* contributors to whether any scheduling format works effectively at producing student achievement.

#### *The Popularity of Shifting to Block Scheduling in the Early 90s*

Block scheduling offers optimism to parents, teachers, and students in that it provides an opportunity for additional learning to occur in ways not possible in a 50- or 55-minute period. Specifically, block scheduling allows students to more deeply examine concepts for extended periods of time. A 4×4 block schedule also allows students to complete eight Carnegie credits in a school year instead of only seven in the traditional schedule. Theoretically, block scheduled classes provide more opportunities for student-centered instruction. Teachers have other positive aspects to consider when teaching under the block schedule format. Each semester, teachers are only preparing for three classes of students each day instead of six in traditional period schedule formats, allowing for more time to prepare and plan. Also, with only three period changes between the four classes, students are in the hallways less frequently, thereby

reducing discipline problems outside the classroom, where teachers have less control.

While, in theory, block scheduling has all the signs of setting up a major reform in the high school classroom, one cannot ignore the negatives that exist in the block-scheduling world. Many classrooms in block format see teachers direct lecturing for 50-55 minutes and then giving students the remaining 30 minutes of class time to do their homework. Or, worse yet, teachers may spend the first 20 minutes of a blocked class going over homework, teach for 45 minutes, and then give students the remaining time to complete homework. In this format, little chance exists for the deep exploration of mathematical concepts or inquiry-based learning that were the foundations behind the creation of block schedule formats. For students, block scheduling can insert a long delay in the cognitive development of critical thinking skills in mathematics. For example, students may take Algebra 1 during the fall block their freshman year. Then, the student might take geometry the following fall during the sophomore year. Essentially, students go nine months without studying mathematics. The same student may then take Algebra 2 in the spring of their junior year. Therefore, students may go two full calendar years without studying algebra extensively. This large time gap in studying algebra in high school may be a serious contributor to the fact that college algebra is the most failed and dropped college course, with calculus closely behind (Adelman, 2003, 2004). We know algebra is a true gatekeeper to access to and attainment in higher education (Jacobson, 2000; Moses, 1994; Rech & Harrington, 2000; Silva & Moses, 1990). The growing popularity of integrated mathematics courses in high school is partially attributed to this notion. Integrating algebra and geometry rather than isolating each subject has been a growing trend in high school mathematics curricula since the mid-90's when the National Science Foundation began funding large projects for creating such curricula.

#### *Existing Policy Worth Examination*

The state of Tennessee recently changed policy regarding the mathematical trajectory that their secondary students will follow during high school. Tennessee now requires all high school students to take mathematics *all four years* of high school—a policy not based on total credits but continuous enrollment (Tennessee Department of Education, 2009). Students are to be evaluated, based on ACT math score, during or immediately following their junior year of high school. Students meeting ACT's recommended achievement level for college mathematics readiness,

an ACT mathematics score of 22 or higher, will have the opportunity to choose which specific course(s) they will take their senior year, dependent on their previous coursework. These choices would consist of Advanced Algebra, Algebra 3, Trigonometry, Pre-calculus, Statistics, and Calculus. Students with an ACT mathematics score of 19 to 21 will be encouraged to take courses focused specifically on preparing students for entry-level college mathematics course or an entry-level job market skills course called *Capstone Mathematics*. Students who have not earned at least a 19 on the ACT mathematics section will be recommended to take *Bridge Math*, a remedial course meant to bring students closer to *college- or job-market readiness*.

Although Tennessee is not alone in wanting more high school students to be college-ready, Tennessee's approach is unique because all students do not have the luxury of avoiding mathematics their senior year. The approach theoretically targets different abilities in an effort to improve students' college readiness based on their mathematics achievement up to the start of their senior year. Michigan already has a similar policy requiring college bound (merit) students to be enrolled in mathematics the senior year without exception (Michigan Department of Education, 2006). Additionally, Kentucky and West Virginia have put policies in place requiring mathematics all four years of high school that should increase mathematics achievement and college-readiness (ECS, n.d.).

Despite these exceptions, many states have a four mathematics course credit policy. While teaching mathematics at a large, mid-Atlantic university in a

state with a four-credit policy in place, I learned how high school students can still avoid *continuous enrollment* in secondary mathematics. Over five years, I taught between 2500 and 3000 students in freshman level mathematics classes. During tutoring, I discovered many of the weaker students avoided continuous enrollment in high school mathematics by taking two mathematics courses in one year under the block schedule format in order to avoid a year of mathematics—usually the senior year. Other weak students had avoided the more advanced mathematics courses in high school by taking half-credit electives in order to fulfill the fourth required mathematics credit. This information was the impetus of this study. I hypothesize that the lack of continual development of critical thinking skills in mathematics resulting from continuous enrollment in secondary mathematics courses diminishes college-readiness.

### Existing Data

#### *Distribution of Scheduling in the U.S.*

Based on NAEP-HSTS<sup>iii</sup> 2005 grade 12 survey data, the 90-minute block is the single most common length of time for classes in today's secondary schools (NAEP data explorer, 2009). In Figure 1, note that traditional 50-minute and 55-minute periods are a close second and third, respectively. However, by a 2-to-1 margin, more classes in the US are less than 60-minutes in length than are 80 minutes in length or longer. Therefore, I must posit the question: If block scheduling is working well, why is it not more common in secondary schools across the United States? Block scheduling has been around for nearly 20 years. If it leads to increases in achievement and college readiness, why have more schools not switched?

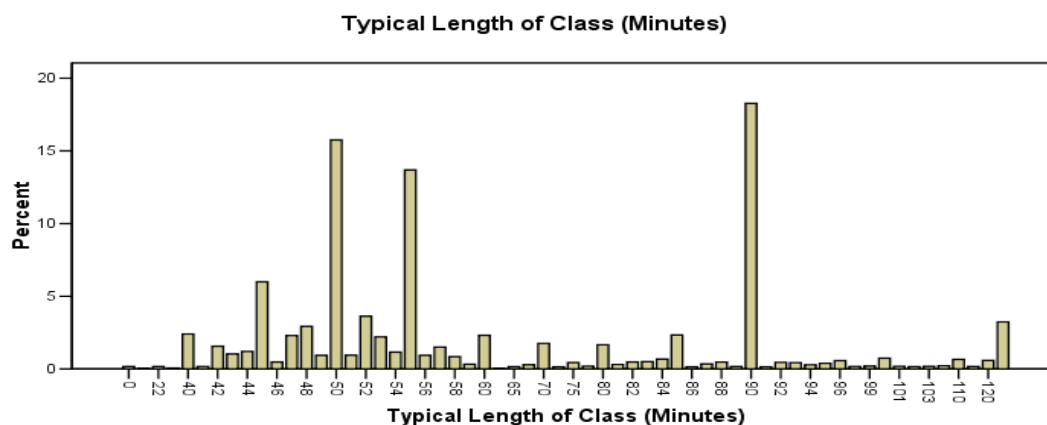


Figure 1. NAEP 2005 12th grade principal survey responses to typical class time length.

*Literature Review on Block Scheduling*

Zepeda and Mayers (2006) analyzed 58 empirical studies concerning block scheduling covering research published from the mid-90s through 2005. However, only a select number of the 58 studies directly researched mathematics achievement. Most studies surveyed, sampled, or tested only small numbers of participants with the exception of Jenkins, Queen, and Algozzine (2002). These researchers studied over 2,100 North Carolina teachers' teaching practices. The data suggested little difference between instructional practices of block and non-block teachers. Moreover, Adams and Salvaterra (1998) reported that although, initially, teachers attempted new progressive teaching strategies in the block schedule format, after a couple years, they seemed to regress to teacher-centered direct lecture without deep inquiry. However, the same research reported that innovative teachers were more positive about block scheduling than teachers who tried to force traditional methods and seatwork into the block format. It is possible to deduce from research that without sustained professional development, teachers regress to less advanced methods of teaching under the block format.

Snyder (1997) reported increased ACT and state test scores with moderate increases in SAT scores for students following block schedules. However, Snyder, as well as Knight et al. (1999), reported that Advanced Placement (AP) exam scores decreased slightly in block format. In contrast, Evans, Tokarczyk, Rice, and McCray (2002) reported that AP scores and standardized test scores increased after students switched from period to block scheduling. Specific to mathematics, Pisapia and Westfall (1997) reported that more schools saw increased SAT verbal scores than increased SAT mathematics scores while all AP exam scores declined. Gruber and Onwuegbuzie (2001) and Lawrence and McPherson (2000) concluded that student achievement across all four major content subjects was better under traditionally scheduled formats than block-scheduled formats. It is clear that the research findings are very mixed regarding the effects of block scheduling on mathematics achievement and college readiness.

*Summary on Block Scheduling from Existing Research*

Zepeda and Mayers (2006) reported three consistent findings in existing research on block scheduling. They claimed the following:

Three major themes across the five groups of studies emerged from the analysis. First, many of the research studies failed to report information

that is customarily found in formal writing such as journal articles and convention papers. Second, the majority of studies, with few exceptions, reported positive perceptions of block scheduling among teachers, students, and administrators. Third, the research presents mixed messages concerning the effect of block scheduling on teachers' instructional practices and on student achievement. (p.158)

The most consistent report on standardized testing, though far from convincing, was that state test scores initially increased under block scheduling. However, these studies were snapshots in time over one year or less. Only one study spanned a longer era and reported initial increases but returned to former achievement levels by year three. Declining AP exam scores and standardized test scores seemed consistent in the few studies that focused on these tests (Cobb, Abate, & Baker, 1999; Knight, De Leon, & Smith, 1999). These findings contradict the strong positive views from students about block scheduling (Hurley, 1997; Salvaterra et al, 1999) and could easily be connected to the fact that grades and grade point average (GPA) increased in block schedule formats (Duel, 1999; Fletcher, 1997; Khazzaka, 1998; Knight et al., 1999; Snyder, 1997). But, these GPA increases may be attributable to a change in grading practices, thus making higher grades easier to obtain, which could, in turn, result in less knowledge and understanding by students.

Zepeda and Mayers (2006) concluded there is good evidence that blocked classes are easier than traditional period scheduled classes because less content is typically covered in blocked classes. In the same six-hour school day, students in a block schedule format complete a full credit more by the end of the year than students in a non-block format, seven Carnegie credits as opposed to eight. Therefore, for any given full credit course, less class time actually exists in the block format to study content. In fact, we can conclude that 12.5% less time is devoted inside the classroom to each course under a block schedule format than in traditional seven period formats because students complete eight full credits instead of seven in one year. Nonetheless, some research does report positive increases in student achievement under a block schedule format. Therefore, this paper examines the three most popular class period lengths with regards to associated mathematics achievement using the national NAEP HSTS 2005 mathematics test data and makes the connection from high school mathematics achievement to post-secondary completion using the NELS 88:2000 data for analysis.

## Methodology

### *Data Sources*

The current two-phase study uses two national data sets. The first phase involved an extensive analysis on the likelihood of bachelor degree completion based on continuous secondary mathematics enrollment using a NELS data set (for a detailed description of NELS see <http://nces.ed.gov/surveys/NELS88/>). Within NELS, only participants who attended four-year post-secondary institutions were analyzed. Further, students who attended highly selective institutions were excluded since advanced high school mathematics courses are required for admission to highly selective institutions. In the NELS:88 dataset, 2.5% of weighted sample population of college students with complete transcripts from higher education attended highly selective institutions. Their removal from the data set prevents skewing the results in favor of the intended research objectives of study. Thus, the population of analysis in NELS removes the highly advanced college bound students and two-year only degree seekers. This remaining population is referred to as the *Typical Bachelor Degree Population* here forward. For comparison purposes, a second analysis was performed on all NELS participants with post-secondary experience (Adelman's [2006] population). This population does not include high school graduates with no post-secondary coursework experience, but does include two-year only degree seekers and the highly selective institution attendees.

The second phase compared mathematics achievement in the three most common (50-55-90 minute classes +/- 1 minute) schedules using the NAEP HSTS 2005 data from the mathematics NAEP test. All NAEP test takers of the mathematics component were included because this part of the analysis examines class format and mathematics achievement rather than college attainment.

### *Predictive Models*

Two predictive statistical models are used in predicting bachelor degree completion using the NELS data. The first model is a logistic regression model controlling for socioeconomic status (SES) and 8th-grade mathematics achievement (prior achievement). This model produces the contribution for each variable to the odds of bachelor degree completion. The dichotomous dependent variable is completion of a bachelor's degree or not. The independent variables considered for the model include entry to Algebra 1 before grade nine, continuous enrollment in secondary mathematics, highest mathematics class completed in

high school, an academic intensity variable constructed by Adelman (2006), and high school mathematics credits required for graduation. The second model uses discriminant analysis (DA). This method is similar to factor analysis. That is, DA allows for examination as to the strength and contribution each variable has towards group separation.

The logistic model may represent one independent variable as contributing to the odds of bachelor degree completion by three times, yet DA may produce results indicating the variable ranks low in separating degree recipients and those who fail to complete their degree. At first glance, this seems to be a contradiction. However, DA puts forth which significant variables are the strongest in separating naturally occurring groups rather than just looking at the odds contribution in logistic modeling. NAEP data compares groups using ANOVA statistics. DA yields results more powerful than the logistic regression models. Thus, type-II errors will be less likely and the degree to which independents separate the two groups will be more easily interpreted in the structure matrix. The structure matrix values are similar to factor loadings and indicate the substantive nature of the independent variables in relationship to each independent's contribution of group separation. Bargman (1970) and Bray and Maxwell (1985) argue that high structure matrix values contribute most to group separation, while lower values contribute least to group separation. The standardized coefficients range from  $\pm 1$  and speak of the relative contribution of each variable in the model. However, the logistic regression models also serve significance while considering all variables in predicting odds of bachelor degree completion. Combined, both models open the door for stronger interpretations of the results. For example, a logistic regression model may indicate the odds of degree completion increases two-fold for a specific variable. Yet, DA may only indicate a small degree of significance by this variable in separating degree earners from non-degree earners. In this example, the variable, say earning a B or better in high school geometry, may indicate a B or better doubles the odds of completing a bachelors degree. In the end, this variable may not be a strong predictor compared to other variables of separating degree and non-degree earner, but still doubles the likelihood of completing a bachelor's degree from those who earn a C or less in geometry.

### *Data Sets Structure, Weights, and Software*

NELS used a sample of U.S. students who were 8th graders in the fall of 1988 and followed these

students through December of 2000. NELS was a two-staged stratified sample of schools and students by probability of random selection. First, schools were randomly chosen. Then, students within the schools were randomly chosen. This complex sampling design requires researchers to meticulously use the weights associated with the research being conducted. The methods suggested for analysis by Thomas and Heck (2001) are used by considering relative weights during analysis with SPSS 17.0. See Thomas, Heck, and Bauer (2005) for an example and detailed explanation.

NAEP is not designed to measure individual student achievement. Therefore, NAEP lists five plausible outcome scores associated with each student in the NAEP restricted data set. The analysis of student achievement in block or traditional scheduled classes uses the existing AM software (v.0.06), which encompasses accepted statistical procedures for using these five plausible values as an outcome measure. Weights are also used in concordance with NAEP and AM software when analyzing achievement in mathematics.

**Statistical Analysis and Results**

*Data Presentation*

The analysis of continuous enrollment using logistic regression is given in Tables 1 and 2. Table 1 presents the findings for the typical bachelor degree seeking population in the US. This group consisted of the students most likely to be influenced if continuous

enrollment in high school mathematics was required over  $x$  number of credits.

All variables considered for analysis, including Adelman’s (2006) predictor variables of academic intensity via Carnegie credits and highest mathematics course completed, are presented in Table 1 alongside three variables of policy importance concerning college-bound students preparing mathematically in secondary schools. There are three results I would like to bring to your attention. First as expected, SES was the predominant labeler of college success for the *Typical Bachelor Degree Population*. Second, early entry to Algebra 1 in eighth grade resulted in decreased odds for degree completion. This may provide some insight into early algebra entry and is discussed further in the interpretation. Third, continuous enrollment did contribute more to increasing the odds of bachelor degree completion than a simple credit increase in mathematics from three to four.

Table 2 presents the results of the analysis for all NELS participants with post-secondary education (PSE) experience (see Appendix for variable coding). When examining the results from the analysis of all students with post-secondary education, two findings emerge when NELS participants with only two-year institution experience are considered. First, SES and highest mathematics completed in high school clearly outweighed all other variables of consideration. Second, graduation credits, continuous enrollment, and early entry to Algebra 1 decrease odds of bachelor degree completion. This is further discussed in the interpretation section.

Table 1

*Analysis of the ‘Typical Bachelor Degree Population’*

Variables Included	$\beta$ (SE)	95% CI for Exp( $\beta$ )		
		Lower	Exp( $\beta$ )	Upper
Constant	-0.298*** (0.053)		0.749	
SES in 12th grade	0.435*** (0.006)	1.527	1.545	1.564
8th grade NELS mathematics score	0.008*** (0.001)	1.007	1.008	1.009
Highest math course completed in HS	0.309*** (0.005)	1.350	1.362	1.374
Overall Academic Intensity	0.004*** (0.001)	1.002	1.004	1.005
Continuous enrollment in secondary mathematics	0.187*** (0.009)	1.185	1.206	1.227
Early entry to algebra	-0.913*** (0.050)	0.364	0.401	0.443
Graduation credit requirements in mathematics	0.047*** (0.006)	1.036	1.048	1.061

Notes:  $R^2=.073$  (Cox & Snell),  $.098$  (Nagelkerke),  $\chi^2(7)=22055.8$ ,  $***p<0.001$ , cases correctly classified=61.3%, and weighted (true)  $N\approx 291K(1.5K)$ .



Table 2

*Analysis of the Full NELS Population with PSE*

Variables Included	$\beta$ (SE)	95% CI for Exp( $\beta$ )		
		Lower	Exp( $\beta$ )	Upper
Constant	-1.377*** (0.028)		0.252	
SES in 12th grade	0.827*** (0.003)	2.273	2.287	2.302
8th grade NELS mathematics score	0.017*** (0.000)	1.016	1.017	1.017
Highest math course completed in HS	0.362*** (0.002)	1.429	1.436	1.442
Overall academic intensity	0.065*** (0.000)	1.066	1.067	1.068
Continuous enrollment in secondary mathematics	-0.237*** (0.005)	0.782	0.789	0.796
Early entry to algebra	-0.843*** (0.025)	0.410	0.431	0.453
Graduation credit requirements in mathematics	-0.061*** (0.003)	0.935	0.941	0.947

Notes:  $R^2=.292$  (Cox & Snell),  $.390$  (Nagelkerke),  $\chi^2(7)=451011.8$ ,  $***p<0.001$ , cases correctly classified=73.6%, and weighted  $N\approx 1.31M$ , true  $N\approx 6.9K$ .

Table 3 and 4 present discriminant analysis results. Results (Table 3 structure matrix) for the analysis of the *Typical Bachelor Degree Population* indicate that the students who advance the most in mathematics can offset low SES or lower prior achievement. However, this cannot be at the expense of an academically intense environment. Most impressive is the fact that increased graduation credits in mathematics do not increase college-readiness, but continuous enrollment

does. When considering all students bound for any form of PSE, only two results stood out: The strength of continuous enrollment nearly tripled, and early entry to algebra decreased slightly. Graduation credits in mathematics remained very weak. SES, advanced mathematics courses, academic intensity, and prior achievement remained relatively the same.

Table 3

*Discriminant Analysis for Both Populations of Study*

Independent Variable	<i>Typical Bachelor Degree Population</i>		All NELS with PSE, Adelman	
	Structure Matrix	Stand. Coeff.	Structure Matrix	Stand. Coeff.
SES in 12th grade	0.581	0.503	0.587	0.433
8th grade NELS mathematics score	0.598	0.126	0.600	0.132
Highest mathematics course completed in HS	0.821	0.734	0.831	0.380
Overall academic intensity	0.603	0.055	0.803	0.384
Continuous enrollment in secondary math	0.157	0.172	0.430	0.092
Early entry to algebra	0.218	0.115	0.126	0.042
Graduation credit requirements in mathematics	-0.025	0.051	0.052	-0.041

Notes: All non-categorical variables mean centered (see Appendix for coding). Respective statistics—Weighted (true)  $N\approx 291K(1.5K)$ ,  $N\approx 1.31M(6.9K)$ , Wilks'  $\lambda=0.927$  &  $0.700$ , & cases correctly classified=61.1% & 73.3%.

Table 4 presents the results by removing two variables and focusing solely on the variables of this study while controlling for SES and prior mathematics achievement. While the overall model strength

weakened slightly from Table 3 to Table 4 (see Wilks’- $\lambda$  and cases classified), continuous enrollment continued to overpower increasing graduation credits in mathematics as a discriminator of college-readiness.

Table 4

*Discriminant Analysis for Both Populations of Study  
(only examining prior achievement, SES, and three variables of this research)*

Independent Variable	Typical Bachelor Degree Population		All NELS with PSE, Adelman	
	Structure Matrix	Stand. Coeff.	Structure Matrix	Stand. Coeff.
SES in 12 <sup>th</sup> grade	0.741	0.639	0.690	0.552
8 <sup>th</sup> grade NELS mathematics score	0.731	0.614	0.708	0.519
Highest mathematics course completed in HS	Removed from analysis			
Overall academic intensity				
Continuous enrollment in secondary math	0.209	0.134	0.529	0.441
Early entry to algebra	0.269	0.196	0.147	0.105
Graduation credit requirements in mathematics	-0.044	0.076	0.066	0.042

Notes: All non-categorical variables mean centered [see Appendix for coding]. Respective statistics—  
Weighted (true) N≈292K(1.5K), N≈1.34M(7.1K), Wilks’- $\lambda$ =0.950 & 0.753, & cases correctly  
classified=59.9% & 71.5%.

*Interpretation of NELS Findings*

Four major findings emerge from the NELS analysis of data. First, SES remains a strong predictor of academic achievement and success. Yet, this variable is not directly controlled by schools or policy. Second, highest mathematics course completed in high school also remained an extremely strong contributor to the odds of bachelor degree completion in the logistic model and to group separation in DA. Yet, when highest mathematics course is removed in the DA, continuous enrollment still outranked additional credit requirements in mathematics for group separation. Third, early entry to Algebra 1 prior to grade nine resulted in mixed findings: Logistic regression indicated decreased odds for degree completion, yet DA provided a positive interpretation. These mixed findings are most likely attributed to the fact that less than 2% of students in NELS took algebra prior to grade nine.

Prior mathematics achievement contributed very little to the odds of degree completion when holding constant at the mean, but DA tells a much more positive result concerning prior achievement in mathematics. So, we must examine the prior

mathematics achievement variable more closely. Prior achievement is a continuous variable with a range of 43 points. A score of 20 points above the mean equates to 1.008<sup>20</sup>. This is interpreted as a 17.3% higher likelihood of completing a bachelor’s degree than an entering high school student with a mean level of achievement. This is an example of how DA can indicate a strong separating variable between groups, whereas logistic regression indicates little likelihood of the outcome variable. We must consider the coding of variables. This last discovery related to prior achievement supports ACT’s contention that we cannot forget policies concerning mathematics education for middle grade college-bound students. Losing students in middle grades mathematics can dramatically shift the likelihood of finishing a four-year degree.

Additionally, model strength was more predictive of bachelor degree completion than non-degree completion. NELS participants who completed bachelor degrees were accurately predicted with the model by 3-to-1, yet non-degree completers were accurately predicted only half of the time. Further, when looking at the two NELS populations (typical and all), it would seem that a universal secondary policy on continuous enrollment is not wise. It does not

appear to benefit non-four-year degree seekers as much as the typical four-year degree seeking student. Thus, continuous enrollment is more appropriate as a college-readiness policy. It is possible continuous enrollment in mathematics may be better implemented in schools and districts for a college-preparatory diploma or curriculum or from higher education institutions through admissions standards, as was recently implemented in Maryland (Maryland Department of Education, 2010).

#### *NAEP Achievement Analysis*

Presented in Table 5 are the results of the analysis of NAEP 2005 12th-grade mathematics achievement data. The NAEP analysis opens some doors to consider

in planning college preparatory scheduling within schools. Under the 4×4 block format, classes are 90-minutes in length. Under the seven-period scheduling format, classes are typically 50-minutes (+/-). The last format typically consists of an abbreviated seven-period day with one or two abbreviated class periods, typically fit into the 55-minute schedule or a trimester schedule. NAEP has indicated that 10 NAEP points equates to one year of academic ability. Essentially, we see the traditional seven-period scheduling 50-minute format equating to nearly two-thirds of a year of academic ability in mathematics achievement over 90-minute block scheduling and 55-minute scheduling.

Table 5

#### *Mean NAEP Mathematics Scores for Dominant Class Scheduling Formats*

Class Time Length	Mean	Weighted N	Std Error (mean)	Std. Dev.	p-value†
49, 50, 51 minutes	156.133	441k	1.745	32.082	
54, 55, 56 minutes	148.637	409k	1.798	33.935	0.003
89, 90, 91 minutes	149.692	432k	2.171	34.300	0.029

†Comparison group is 49-50-51. 55 vs. 90 not significantly different.

#### *Cautions in Interpretation*

Based on the literature review by Zepeda and Mayers (2006), this data should be interpreted with caution. Schools that follow block schedules may do so merely to combat discipline problems of underachieving students. Therefore, this practice may explain the differences between the student NAEP scores under the 50- and 90-minute schedules. There is little question that school characteristics and student socioeconomic status are factors in the findings. Moreover, Zepeda and Mayers's summary of research suggests that teacher characteristics also play a role in determining mathematics achievement. Thus, we can only tentatively say that 50-minute class periods produce significantly higher mathematics achievement than the 55- and 90-minute options. This preliminary analysis introduces another unanswered question: Why is there such a difference between 50- and 55-minute classrooms? Clearly, a multilevel analysis is warranted examining school-level factors.

#### **Implications**

##### *Continuous enrollment in secondary mathematics*

NAEP sheds some light on the hypothesis that continuous enrollment may increase mathematics achievement and the development of critical thinking skills. This analysis supports the hypothesis that continuous enrollment in secondary mathematics may be more likely to produce a *college-ready* student over a *college-eligible* student. Students enrolled in 90-minute semester-blocked classes scored, on average, two-thirds of an academic year of ability in mathematics lower than students enrolled in 50-minute year-long mathematics courses. Further studies are needed to explore the reasons why 55-minute classes are similar in achievement to 90-minute classes. School level factors need to be examined more deeply.

NELS data analysis tells a similar story. Continuous enrollment is a much stronger predictor of bachelor degree earners than increasing mathematics credits alone for graduation from high school. Continuous enrollment increased the odds of bachelor degree completion with logistic regression and with discriminant analysis. Mathematics credits for graduation alone was the weakest predictor variable

but significant. NELS is from a time where block period schedules were virtually absent from all high schools in America. The Educational Longitudinal Study (ELS), the current NCES longitudinal study, will shed much more light on college readiness related to secondary mathematics and scheduling formats. That study will be completed in 2012 or 2013 at the earliest for post-secondary attainment examination.

### *Changing Mathematics to be Socially Acceptable*

In the US it is acceptable to say “*I am not good at math*”, “*I hate math*” or “*Don’t worry honey [parent referring to child], I was not good at math either, so it is ok for you too.*” However, you will rarely, if ever, hear someone admit in our society that they cannot read. We would be embarrassed if we could not read, but we are eager at times to admit poor mathematics skills and understanding. If we want society to begin to shift away from the socially accepted norms regarding mathematics and embrace the critical thinking skills that can be enhanced in the mathematics classroom, then we need to parallel policies for English, literacy, and reading. If we require nearly all U.S. high school students to be enrolled continuously in Language Arts, then should we not be doing the same with mathematics? Rather than focus on accumulated credits, we should focus on the continuous development of critical thinking skills through challenging mathematics classes. Students may revolt from fear of a lowered GPA. Parents and teachers may also dislike such a policy. However, the data presented in this paper tells a significant story in support of this position. In the United States, secondary mathematics achievement, college-readiness, and bachelor degree completion rates are not at acceptable levels. This research supports the contention that continuous enrollment in secondary mathematics increases mathematics achievement and the likelihood of bachelor degree completion more so than does the accumulation of high school mathematics credits.

This research paper supports the notion that mathematics in secondary schools, in preparing tomorrow’s college bound students, should parallel English language arts in the continuous development of critical thinking skills through rigorous coursework during all four years of high school. In the future, research is planned using NAEP in a multilevel fashion where school and student level characteristics are considered. NELS and ELS (once completed in 2012-2013) comparisons will provide insight as to how block schedules have changed college readiness for the better or worse over the last two decades.

<sup>i</sup> Four-by-four block scheduling refers to the school format of classes where students take a full Carnegie credit in half the school year (semester) typically offered as daily 90-minute classes. Students can complete eight Carnegie credits in one full academic school year. Traditionally, before block scheduling appeared in United States schools, full Carnegie credit courses were completed during an entire school year in daily 45-60 minute classes, referred to as traditional period scheduling formats.

<sup>ii</sup> ACT, Inc., formerly America College Testing, is one of two predominant “college-readiness” or “college-predictive” tests used in the United States for college entrance. SAT, or the Scholastic Aptitude or Assessment Test, is the other.

<sup>iii</sup> NAEP-HSTS 2005—National Assessment of Education Progress High School Transcript Study of 2005.

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### Appendix

- Student SES @ 12<sup>th</sup> grade – NELS variable code F2SES3, mean-centered
- 8<sup>th</sup> grade math achievement – NELS variable code BY2XMSTD, mean-centered
- Overall academic intensity – NELS PETS code ACLEVEL, reverse coded low to high
- High math – NELS PETS code HIGH MATH, coded 6-calculus, 5-precalc, 4-trig, 3-algebra2, 2-geometry, 1-algebra1, 0-below algebra1
- Continuous enrollment in math – Coded “1” using transcript data if student received a letter grade (A,B,C,D,E,F) each term of high school on transcript. W or no grade coded “0”
- Early entry to algebra-1 – Coded “1” if transcript data indicated letter grade received prior to grade nine in algebra-1
- Graduation credits in math – NELS variable code F1C70B, recoded 4=4yrs, 3=3yrs, 2=2yrs, 1=1yr, 0=<1yr, -1=none

# Reading Grade Levels and Mathematics Assessment: An Analysis of Texas Mathematics Assessment Items and Their Reading Difficulty<sup>1</sup>

John H. Lamb

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Increased reading difficulty of mathematics assessment items has been shown to negatively affect student performance. The advent of high-stakes testing, which has serious ramifications for students' futures and teachers' careers, necessitates analysis of reading difficulty on state assessment items and student performance on those items. Using analysis of covariance, this study analyzed the effects of reading grade level of mathematics assessment items on student performance on the Texas Assessment of Knowledge and Skills. Results indicated that elementary and middle school students performed significantly worse on mathematics assessment items having a reading level above the student grade level. The implications of these results are discussed.

Rubenstein (2000) stated, “[Teachers of mathematics] want [students] to speak the language of mathematics, using standard terms that others recognize and understand” (p. 243). Boero, Couek, and Ferrari (2008) noted that the language of mathematics requires a mastery of one’s natural language, both words and structures, in order to incorporate this language within the context of mathematical syntax. They illustrated how teachers of mathematics must go beyond simple classroom discussions in order to promote their students’ mastery of the language of mathematics.

Carter and Dean (2006) illustrated how 5th through 11th grade mathematics teachers spend a considerable amount of time teaching reading strategies such as decoding written language into spoken words, understanding and discovering the meaning of new vocabulary, and making connections between the written language and the learner’s prior knowledge. Out of 72 mathematics lessons they observed, nearly 70% of the implemented reading strategies addressed vocabulary. According to Lager (2006), “Without a strong command of both everyday language and specialized mathematical language students cannot fully access the mathematics content of the text, lesson, or assessment” (p. 194). Teachers of mathematics are faced with the challenge to not only prepare their students to successfully understand mathematical

concepts but to also to prepare students to read and comprehend technically dense, descriptive mathematics problems.

## Student Difficulties in Reading Related to Mathematics Achievement

Evidence has shown that a student’s level of reading proficiency can be a strong indicator of mathematical success (Jiban & Deno, 2007). The correlation between reading and mathematics achievement has been well documented over the last five decades (e.g., Breen, Lehman, & Carlson, 1984; Fuchs, Fuchs, Eaton, Hamlett, & Karns, 2000; Helwig, Rozek-Tedesco, Tindal, Heath, & Almond, 1999; Jerman & Mirman, 1974; Pitts, 1952; Thompson, 1967). Evidence suggests that students without a strong ability to read and with difficulties in mathematics struggle more to be as successful as they could be in mathematics when compared to students only having difficulties in mathematics (Jordan, Hanich, & Kaplan, 2003). This situation is exacerbated for students with limited English proficiency (LEP).

Many students with LEP have accommodations to help account for students’ academic difficulties related to their language deficiency. Accommodations may include extended time to complete assessments or

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assignments and having test questions read aloud. In Oregon, 6th-grade students with low reading ability performed better on mathematical problem solving assessment items when each assessment item was projected on a video monitor while a recorded narration of the written portions of the assessment items was played (Helwig et al., 1999). Fuchs et al. (2000) conducted a quasi-experimental study that randomly assigned students with and without learning disabilities (LD) to treatment accommodations that included extended time, calculator use, having questions read aloud, and encoding (when the teacher writes the student's responses). Fuchs et al. found these accommodations significantly benefited students with LD on their achievement in mathematical problem solving. Ketterlin-Geller, Yovanoff, and Tindall (2007) found that students with lower reading abilities scored better on linguistically and mathematically difficult assessment items when the questions were read aloud. Bolt and Thurlow (2007) found that reading questions aloud to students had a positive effect on 4th-grade student performance on mathematical assessment items with challenging text. For 8th-grade students, however, this positive effect was not evident. As this study demonstrates, there are inconsistencies in research conclusions that both reinforce and challenge the validity of such accommodations (Ketterlin-Geller, Yovanoff, & Tindal, 2007).

#### *Reading Difficulty of Assessment Items Related to Mathematics*

When students have difficulties in reading, research has indicated that they also struggle in mathematics. The way mathematics assessment items are written may influence the mathematical achievement of these poor readers. In 1967, Thompson generated two sets of mathematically similar test items written at different reading levels. He found that sixth grade students performed significantly better on the assessment items written at a lower reading level. Jerman and Mirman (1974) correlated several measures of readability with student performance on mathematics assessment items and found that the higher the word count, character count, sentence count, syllable count, word length, and sentence length of an assessment item, the lower the student performance. Similarly, Walker, Zhang, and Surber (2008) found that the reading difficulty of mathematics assessment items significantly lowered student performance. Hence, research conducted in recent decades has indicated mathematics assessment items and how they are written can have an effect on student performance. One may question how connected these two subjects

are in terms of their predictive power towards one another.

#### *Predictive Power of Reading Ability on Mathematics Achievement*

Research indicates that students with difficulties in reading tend to have lower achievement in mathematics (e.g., Fuchs et al., 2000; Jiban & Deno, 2007; Jordan et al., 2003; Reikerås, 2006). This supports the research indicating a strong correlation between reading and mathematics achievement (e.g., Breen, Lehman, & Carlson, 1984; Pitts, 1952). Evidence also exists for a connection between higher reading levels of mathematics assessment items and decreased student performance (e.g., Jerman & Mirman, 1974; Thompson, 1967; Walker et al., 2008). With the available data indicating reading affects mathematical performance, several states have implemented quantitative measures that utilize students' reading scores as predictive variables in their mathematics performance. The push for these states to identify predictive models came from the United States Department of Education (USDOE) Growth Model Pilot Program initiated in 2005 (USDOE, 2008). Twenty-three states have submitted applications, and as of January 2009, 15 states, including Texas, have been fully or conditionally approved (USDOE, 2009). The Texas Education Agency (TEA, 2009) established a Texas Projection Measure (TPM) that generated student expectant scores on state mathematics assessments based on reading and mathematics scores from previous end-of-year assessments. The TPM is designed to strengthen TEA's measure of Adequate Yearly Progress (AYP) for the No Child Left Behind (NCLB) law by generating a measure that assesses growth. This growth is evaluated using predictive measures for each child's 5th-, 8th-, and 11th-grade achievement level. The TPM projects a student's performance in 5th-grade reading and mathematics by utilizing 3rd- and 4th-grade reading and mathematics performance results. The TPM predicts 8th-grade performance by using 5th-, 6th-, and 7th-grade results and 11th-grade performance by using 8th-, 9th-, and 10th-grade results. TEA found that nearly 1% of the variance in mathematics was accounted for by the student's previous reading performance and up to 5.2% of the variance in reading was accounted for by the student's previous mathematics performance. TEA generated these results by finding the difference in explained variance with and without the other subject as a predictive variable. TEA showed that this percentage of explained variance was significant and

highlights the likelihood that mathematics and reading abilities are highly intertwined.

The Ohio Department of Education (ODE, 2007), also using a USDOE-approved growth-model with a regression-based value-added design of assessing student growth, found results similar to TEA (2009), identifying strong correlations and covariance between reading and mathematics achievement. In 2003, nearly two years prior to the USDOE's growth-model pilot program, Mississippi proposed and implemented an accountability model that predicted future performance in mathematics based on students' previous English, reading, and mathematics results on state assessments (Mississippi Department of Education, 2008). States across the country have illustrated, both in action and through quantitative measures, that reading and mathematics achievement are connected.

### Purpose of This Study

The purpose of this study was to determine whether the reading grade level (RGL) of mathematics assessment items had a significant effect on 3rd-through 11th-grade student performance in the state of Texas. RGL of a mathematics assessment item was operationally defined as the approximate grade level a student was expected to obtain in order to comprehend a reading passage within the item. Based on the available research, I hypothesized that student performance is negatively affected by the RGL of mathematics assessment items on the state mandated Texas Assessment of Knowledge and Skills (TAKS).

To test this hypothesis, I analyzed assessment items and utilized available item analysis data from the 2006 TAKS. Several extraneous variables were

identified and used as covariates in order to isolate the effect RGL might have on student performance. The three research questions in this study were:

1. Do students in grades 3 through 11 perform better on mathematics assessment items that are written with a RGL At-or-Below their grade level than on assessment items written with a RGL above their grade level?
2. Do students in specific grade bands perform better on mathematics assessment items that are written with a RGL At-or-Below their grade level than on assessment items written with a RGL above their grade level?
3. Do students regardless of their grade or grade band perform better on mathematics assessment items that are written with a RGL At-or-Below their grade level than on assessment items written with a RGL above their grade level?

### Methods

This study used the Spring 2006 TAKS 3rd-11th grade released tests and item analysis from TEA (2008a). The TAKS objective and percentage of Texas students who correctly responded for each assessment item was collected for 438 test items. Of the six TAKS objectives in items for grades 3 through 8, five were aligned with the NCTM (2000) content standards, and the sixth objective was aligned to the NCTM process standard, problem solving. Ten 9th through 11th grade TAKS objectives incorporated algebraic, geometric, and problem solving objectives. Tables 1 and 2 list the total number of TAKS assessment items at each grade level and categorize each grade-level objective.

Table 1

#### *Texas Assessment of Knowledge and Skills Objectives at Grades 3-8*

Objective	Number of Items per Grade Level					
	3rd	4th	5th	6th	7th	8th
1—Numbers, operations, and quantitative reasoning	10	11	11	10	10	10
2—Patterns, relationships, and algebraic reasoning	6	7	7	9	10	10
3—Geometry and spatial reasoning	6	6	7	7	7	7
4—Measurement	6	6	7	5	5	5
5—Probability and statistics	4	4	4	6	7	8
6—Mathematical processes and tools	8	8	8	9	9	10
Total Number of Items	40	42	44	46	48	50

Table 2

*Texas Assessment of Knowledge and Skills Objectives at Grades 9-11*

Objective	Number of Items per Grade Level		
	9th	10th	11th
1—Functional relationships	5	5	5
2—Properties and attributes of functions	5	5	5
3—Linear functions	5	5	5
4—Linear equations and inequalities	5	5	5
5—Quadratic and other nonlinear functions	4	5	5
6—Geometric relationships and spatial reasoning	4	5	7
7—2D and 3D representations	4	5	7
8—Measurement	6	7	7
9—Percents, proportions, probability, and statistics	5	5	5
10—Mathematical processes and tools	9	9	9
Total Number of Items	52	56	60

*Reading Grade Level*

Prior to conducting this study, I needed to determine if TEA relied on any measure of reading grade level when creating assessment items for the TAKS. In each of the TEA (2008b) TAKS Information Booklets, a passage of text was used to illustrate how experts reviewed and determined the appropriateness of each assessment item prior to field testing. TEA did not specifically state whether experts analyzed the reading level of each mathematics assessment item; however, the use of experts in education, specifically mathematics education, does provide some level of validity that items were written at appropriate reading grade levels. Because TEA did not quantify the reading level of TAKS assessment items, a quantitative measure of the readability of these assessment items needed to be determined for the purposes of this study.

In 1935, Gray and Leary found that there were nearly 228 variables that affected readability. Semantic, syntactic, and stylistic elements accounted for the majority of these variables. Over the last century, numerous methods have been created to determine the readability of a passage. In 1948, Rudolph Flesch created one of the earliest formulas, the Flesch Reading Ease Formula. This formula incorporated the variables of sentence length and syllable count into a calculation of reading difficulty. Working for the United States Navy, Kincaid, Fishurn, Rogers, and Chissom (1975) adapted the Flesch Reading Ease Formula, known as the Flesch-Kincaid Grade Level Formula, to provide an outcome that better predicts a reader's grade-level. Edward Fry

(1977) further advanced the science of readability by creating another popular readability formula known as the Fry Graph Readability Formula. He utilized the averages of sentence count and syllable count per 100 words to determine an ordered pair (average sentence count, average syllable count) located within sectors of a coordinate plane corresponding to the reading age of the text. The Flesch Reading Ease Formula, Flesch Kincaid Grade Level Formula, and Fry Graph Readability Formula all utilized some type of average counts based on sentences, words, and syllables that provided strong support for the syntactic structure of a passage of text and an effective means for determining semantic complexity. Determining readability based on these methods did not improve until the use of electronic databases surfaced in recent decades.

Electronic databases have been useful in improving the validity and reliability of calculated semantic complexity in readability formulas. The New Dale-Chall Readability Formula incorporates the variable of sentence length and a calculated percentage of words not found in a database of 3,000 common 4th-grade vocabulary terms (Chall & Dale, 1995). Chall and Dale believed words not found on this list of 3,000 common words were more difficult and thus used a calculated percentage of words not found in this list as a measure to produce higher grade-equivalence scores. Touchstone Applied Science Associates (1999) built upon the work of Chall and Dale (1995) by using a database of common vocabulary terms in their readability formula and by adding a new calculation of the average number of letters per word.

One of the latest and most sophisticated measures of readability is the Lexile Framework © for Reading (referred to as “Lexile”) (2008). Lexile measures of readability are also based on semantic complexity and sentence length, but Lexile determines semantic complexity through the calculation of word frequency incorporating a database of nearly 600-million words, whereas other databases have only 5- to 25-million words (Wilson, Archer, & Rayson, 2006). Lexile has incorporated the advances of the past and paired them with an enormous database of words.

Lexile text measures are based on two well-established predictors of how difficult a text is to comprehend: semantic difficulty (word frequency) and syntactic complexity (sentence length). In order to determine the Lexile measure of a book or article, the text is split into 125-word slices. Each slice is compared to the nearly 600-million word Lexile corpus—taken from a variety of sources and genres—and the words in each sentence are counted. These calculations are put into the Lexile equation. Then, each slice’s resulting Lexile measure is applied to the Rasch psychometric model to determine the Lexile measure for the entire text. (Lexile Frequently Asked Questions, 2008)

The Lexile measure was chosen for this study because it is a grade equivalence measure that accounts for both syntactic and semantic elements, relies on an extensive database, and is readily accessible.

The first step in calculating the Lexile (2008) Measure for the TAKS assessment items required each mathematics assessment item to be converted into a text format, thus eliminating any graphs, tables, or other figures not representing standard sentence structure. Next, each assessment item was uploaded into the online Lexile Analyzer to obtain a Lexile Measure ranging from 10L to above 1700L. Each Lexile Measure was then located on the Lexile Map to determine its approximate reading grade level (RGL) ranging from 1 to 17, where scores of 13 through 17 represent post-secondary grade equivalencies. The Lexile Map provides an interval of Lexile Measures that correspond to a particular grade level at which the reader should have at least a 75% comprehension. Many Lexile Measures span more than one grade level. For instance, 3rd-grade Lexile Measures range from 520L-750L and 4th-grade Lexile Measures range from 620L-910L. In this case of overlapping Lexile Measures, 0.5 was added to the lowest grade level approximation. Therefore, in this study, a Lexile

Measure of 620L would be assigned a grade equivalence of 3.5.

### *Cognitive Demand*

According to the Wisconsin Center for Educational Research (WCER, 2008), there are five cognitive demand categories for mathematics: (1) memorize; (2) perform procedures; (3) demonstrate understanding; (4) conjecture, prove, solve; and (5) apply/make connections. The researcher used his training provided by the WCER to train nine groups of K-12 mathematics teachers on how to rate assessment items based on these cognitive demand categories. Grade-level groups of three to five teachers rated TAKS mathematics assessment items. After teachers rated the items individually, they discussed their ratings with their group. Because some assessment items could address more than one cognitive demand category, the teachers could categorize an item in up to three cognitive demand categories. An overall average cognitive demand score based on the teachers’ categorizations was computed for each assessment item.

In general, the cognitive demand of each TAKS assessment had an even distribution of the cognitive demand levels from 1 to 5. However, as illustrated in Figure 1, the cognitive demand ratings for the 5th- and 11th-grade assessments did not match this trend. The 5th-grade distribution of cognitive demand levels had a much smaller range of scores and included more outliers than any of the other grade levels. The 11th-grade interquartile range of cognitive demand levels was elevated as compared to the other grade levels. In both cases, the unusual distribution may be the result of a single outspoken teacher in each group. Future research may use other means to categorize the cognitive demand level of assessment items to minimize this issue.

### **Data Analysis**

An analysis of covariance (ANCOVA) determined differences in student performance based on the between-subjects factor of RGL. On a mathematics assessment, the item’s content strand and level of cognitive demand are known variables that influence student performance. Field (2009) noted that when variables not part of the main experiment have an influence on the dependent variable, then an ANCOVA should be used to control for the effect of these covariates. Field also explained that two assumptions in an ANCOVA should be tested: “(1) independence of

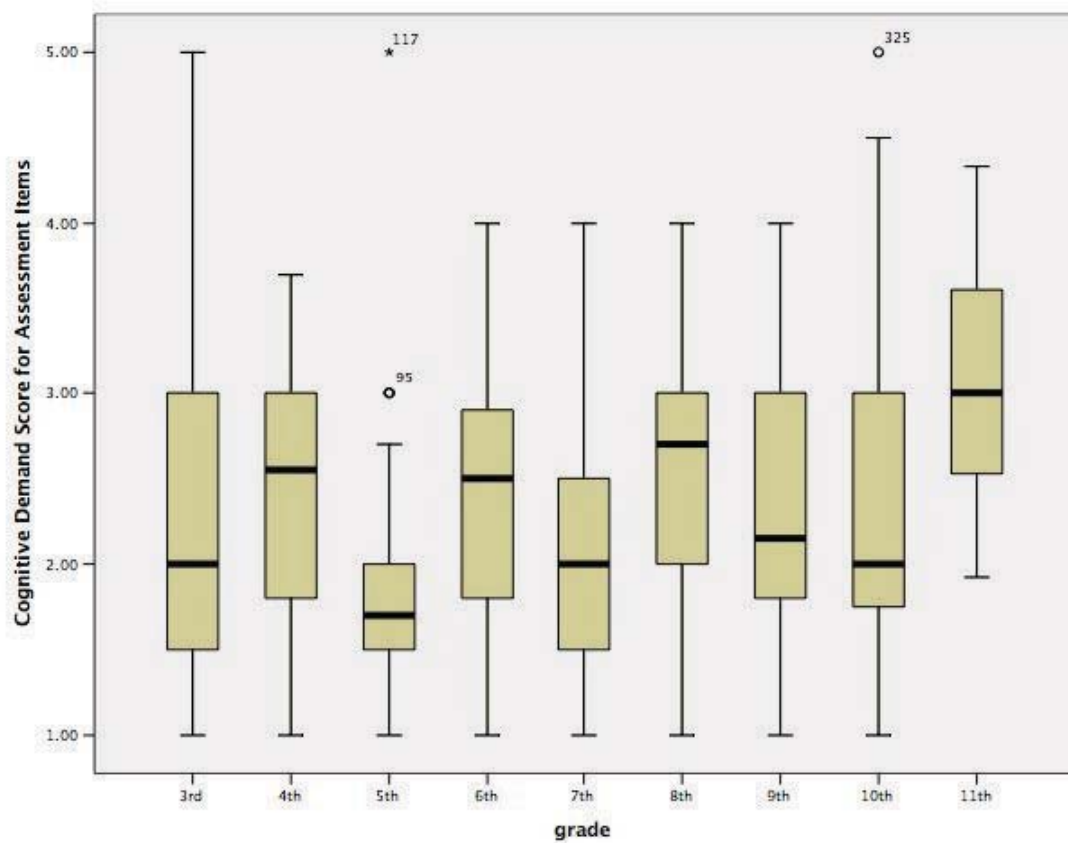


Figure 1. Box plots of cognitive demand scores for each TAKS grade level

the covariate and treatment effect, and (2) homogeneity of regression slopes” (p. 397). For these reasons, the data analysis in this study included tests of independence and homogeneity.

The mathematics TAKS assessment items were categorized based on whether the item’s RGL is At-or-Below grade level (henceforth written as RGL At-or-Below) or above the student’s grade level (henceforth written as RGL Above), establishing the independent variables in this study. For example, a 3rd-grade TAKS mathematics assessment item having a 2.5 RGL would be categorized as RGL At-or-Below, and a 3rd grade TAKS assessment item having a 5.0 RGL would be categorized as RGL Above. The dependent variable in this study was the percentage of students who correctly answered each assessment item.

An ANCOVA was conducted at three levels. The first level of analysis was a 2 x 9 factorial ANCOVA with RGL as the between-subjects factor and student grade level as the within-subjects factor. The

covariates were cognitive demand and TAKS objective of each assessment item. The second level of analysis was a 2 x 3 factorial ANCOVA with RGL as the between-subjects factor and the assessment item’s grade band (i.e., elementary school grades 3-5, middle school grades 6-8, or high school grades 9-11) as the within-subjects factor. The covariates of student grade level, cognitive demand, and TAKS objective data were used in this analysis. The final level of analysis grouped all assessment items from each TAKS grade level together. At this level, the covariates were grade level of each assessment item and the item’s cognitive demand. SPSS 13 © was used for all statistical analysis in this study.

Tests of independence and homogeneity were also conducted for each ANCOVA, at each level of analysis. The test of independence consisted of a univariate analysis to determine if the RGL groups differed on each covariate. If no significant difference occurred between the RGL groups based on the

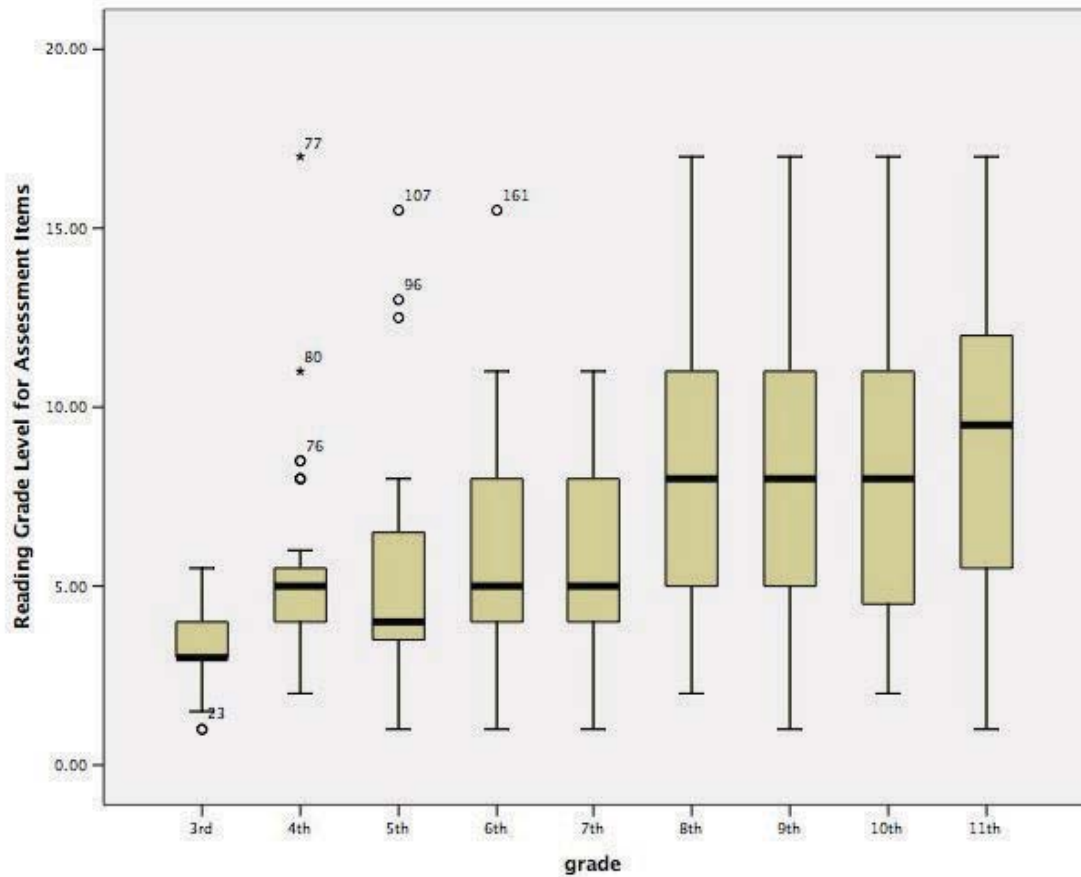


Figure 2. Box plot of the Reading Grade Level based on TAKS grade level

covariate, then the covariate was used in the ANCOVA. The test of homogeneity occurred after each ANCOVA with the researcher running the test a second time to check for any significant interactions between the independent variable of RGL and each covariate. If significant interaction was observed, the researcher further investigated this interaction using a univariate analysis between the RGL groups and covariate to further explain the significant interaction and determine effect (Tabachnick & Fidell, 1996).

**Results**

The average RGL for mathematics assessment items at each TAKS grade level ranged from 3.15 to 9.08 (see Table 3). All three elementary grades (3–5) had an average RGL greater than the student grade level being assessed. Excluding the 8th grade, all other grades had an average RGL below the student grade level being assessed.

As illustrated in Figure 2 and Table 3, the variance in RGL increased with the increase in the grade level. This increase in variance for higher grade levels was to be expected given that higher grade level assessments have a larger range of possible RGLs. It was also expected and confirmed that, at each grade level, there were assessment items written at a very low RGL. These assessment items were the standard mathematics questions requiring students to simply evaluate expressions or solve problems having no words other than action verbs and phrases like *solve*, *add*, *find the product*, or *evaluate*. Although there was a gradual increase in the range of RGL at each grade level, five out of the seven grade levels had assessment items with RGL measures at the graduate level (>16). Only the third grade assessment items were found to have a reasonable range of RGL measures, from 1 to 5.5.

The outliers identified in the cognitive demand and reading grade level variables presented some concern. Initial analysis at all three levels included assessment item data that were outliers in either RGL or cognitive

demand. Analysis was performed again, at all three levels, with these assessment items removed. At all three levels of analysis, there were no differences in results between the data with and without outliers. Therefore, in this study, the initial analysis results, using data that included outliers, is presented.

Table 3

*Reading Grade Level Descriptive Statistics*

TAKS Grade Level	Min. RGL	Max. RGL	Mean	Std. Dev.
3rd	1	5.5	3.15	0.99
4th	2	17	5.21	2.64
5th	1	155	5.23	2.89
6th	1	15.5	5.88	3.03
7th	1	11	5.81	2.58
8th	2	17	8.04	3.85
9th	1	17	8.15	4.03
10th	2	17	8.15	4.31
11th	1	17	9.08	4.18

A mean comparison of student performance based on the between-subjects factor of RGL was conducted at each grade level, each grade band, and over the entire set of mathematics assessment items. Items with an RGL At-or-Below had a higher percentage of successful students than items with an RGL Above in every grade level, grade band, and over the entire set of items except for 9th and 11th grade assessment items (see Table 5). These mean comparisons, however, do not account for the variance explained by other confounding variables. Therefore an analysis of covariance helped to determine differences in student performance at each grade level, grade band, and overall assessment items. The adjusted marginal means are provided in Table 5 to illustrate differences after controlling for various covariates. Inspection of differences between RGL group means, using either adjusted or unadjusted means, showed the 7th-grade RGL At-or-Below student performance average to be over 10 percentage points higher than the average for the 7th grade RGL Above. This adjusted mean difference between the two groups of items at the 7th grade level was the largest at any grade level, grade band, or over the entire set of assessment items. This observable difference at the 7th grade is discussed in great detail below.

*Student Grade Level Analysis*

The initial test of independence between the RGL groups and covariates of TAKS objective and cognitive demand resulted in no significant differences. This allowed for the inclusion of both covariates in the ANCOVA. Table 4 provides results from the ANCOVA performed at each grade level. It indicated a significant difference between RGL At-or-Below items and RGL Above items at the 7th-grade level ( $F(1, 44) = 8.336$ ,  $p = 0.006$ ,  $\eta^2 = 0.159$ , observed power = 0.81). Tests of homogeneity between the RGL groups and the covariates of cognitive demand and TAKS objective had no significant interactions except at the 10th-grade level. In this case, homogeneity was violated between the RGL groups and cognitive demand ( $F(2, 51) = 6.453$ ,  $p = 0.003$ ). Further investigation into this significant interaction revealed no significant difference in cognitive demand between the 10th-grade RGL groups ( $F(1, 54) = 1.300$ ,  $p = 0.259$ ). Therefore, the inclusion of the covariate in the analysis is justified (Tabachnick & Fidell, 1996).

Table 4

*Tests of Between-Subjects Effects of RGL at Each Grade Level*

Grade	F	Sig.	$\eta^2$	Observed Power
3rd	0.752	0.391	0.020	0.135
4th	1.778	0.190	0.045	0.255
5th	3.708	0.061	0.085	0.468
6th	2.571	0.116	0.058	0.347
7th	8.336	0.006	0.159	0.806
8th	0.060	0.808	0.001	0.057
9th	0.149	0.701	0.003	0.067
10th	2.788	0.101	0.051	0.374
11th	0.011	0.917	0.000	0.051

Note:  $\alpha = 0.05$

*Student Grade Band Analysis*

The second phase of analysis of covariance yielded results related to differences in student performance at each of the three grade bands. The initial test for independence found that the covariates of cognitive demand, TAKS objective, and student grade level did not have any significant differences between the RGL

Table 5  
*Adjusted and Unadjusted Student Performance Means and Variability*

Grade Level	RGL	N	Unadjusted		Adjusted	
			M	SD	M	SE
3rd	At-or-Below	28	81.04	11.12	81.10	2.42
	Above	12	77.42	17.44	77.26	3.70
	Total	40	79.95	13.20	79.18	2.20
4th	At-or-Below	20	82.45	7.79	82.51	1.60
	Above	22	79.59	7.78	79.54	1.53
	Total	42	80.95	7.82	81.02	1.10
5th	At-or-Below	30	82.27	6.57	82.31	1.31
	Above	14	77.86	7.93	77.77	1.93
	Total	44	80.86	7.24	80.04	1.15
6th	At-or-Below	33	78.03	11.03	77.98	2.06
	Above	13	71.62	13.06	71.75	3.29
	Total	46	76.22	11.85	74.86	1.94
7th	At-or-Below	34	<b>70.29</b>	9.95	<b>70.87</b>	1.90
	Above	14	<b>61.71</b>	13.74	<b>60.31</b>	3.03
	Total	48	67.79	11.72	65.59	1.75
8th	At-or-Below	31	67.97	12.96	67.74	2.25
	Above	19	66.47	11.10	66.84	2.89
	Total	50	67.40	12.19	67.29	1.82
9th	At-or-Below	32	61.03	12.97	61.34	2.41
	Above	20	63.35	14.49	62.86	3.06
	Total	52	61.92	13.48	62.10	1.92
10th	At-or-Below	40	66.25	13.25	65.49	1.87
	Above	16	57.56	11.06	59.47	3.01
	Total	56	63.77	13.18	62.48	1.75
11th	At-or-Below	43	40.86	14.46	40.86	2.20
	Above	17	41.29	15.06	41.29	3.51
	Total	60	40.98	14.50	41.08	2.07
Elementary	At-or-Below	78	81.87	8.66	81.96	1.04
	Above	48	78.54	10.80	78.40	1.32
	Total	126	80.60	9.63	80.18	0.84
Middle School	At-or-Below	98	72.16	12.01	71.95	1.20
	Above	46	66.48	12.81	66.94	1.76
	Total	144	70.35	12.51	69.44	1.06
High School	At-or-Below	115	55.30	17.68	55.65	1.37
	Above	53	54.53	16.49	53.78	2.03
	Total	168	55.06	17.27	54.71	1.22
Overall	At-or-Below	291	68.10	17.68	68.53	0.77
	Above	147	66.11	16.86	65.28	1.09
	Total	438	67.43	17.42	66.90	0.67

groups at the elementary and high school levels. However, a significant difference was found between the RGL groups based on the cognitive demand covariate at the middle school level ( $F(1, 142) =$

$5.540, p = 0.020$ ). This led to the rejection of the assumption that the covariate and treatment effect were independent, and therefore the cognitive demand variable was omitted from the ANCOVA for the



Table 6

*Tests of Between-Subjects Effects of RGL at each Grade Band*

Grade Band	Source	Type III Sum of Squares	df	Mean Square	F	Sig.	$\eta^2$	Obs. Pwr. <sup>(a)</sup>
Elementary School	Cognitive Demand	22.126	1	22.13	0.264	0.608	0.00	0.08
	TAKS Objective	1097.001	1	1097.00	13.107	0.000	0.10	0.95
	Student Grade Level	23.005	1	23.01	0.275	0.601	0.00	0.08
	RGL	374.203	1	374.20	4.471	0.037	0.04	0.56
	Error	10127.313	121	83.70				
Middle School	TAKS Objective	24.382	1	24.382	0.173	0.678	0.00	0.07
	Student Grade Level	1612.186	1	1612.186	11.429	0.001	0.08	0.92
	RGL	773.341	1	773.341	5.482	0.021	0.04	0.64
	Error	19747.948	140	141.057				
High School	Cognitive Demand	1108.06	1	1108.06	5.139	0.025	0.03	0.62
	TAKS Objective	455.103	1	455.10	2.111	0.148	0.01	0.30
	Student Grade Level	14089.68	1	14089.68	65.34	0.000	0.29	1.00
	RGL	125.434	1	125.43	0.582	0.447	0.00	0.12
	Error	35148.711	163	215.64				

<sup>(a)</sup> Computed using  $\alpha = 0.05$

Table 7

*Tests of Between-Subjects Effects of RGL Over All Assessment Items*

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	$\eta^2$	Obs. Pwr. <sup>(a)</sup>
Cognitive Demand	0.025	1	0.025	0.000	0.990	0.00	0.05
Student Grade Level	54532.535	1	54532.535	315.162	0.000	0.42	1.00
RGL	1025.366	1	1025.366	5.926	0.015	0.01	0.68
Error	75095.139	434	173.030				

<sup>(a)</sup> Computed using  $\alpha = 0.05$

middle school grade band. The ANCOVA yielded significant differences between the RGL At-or-Below assessment items and RGL Above assessment items in both the elementary ( $F(1, 121) = 4.471, p = 0.037, \eta^2 = 0.036$ , observed power = 0.56) and middle school ( $F(1, 140) = 5.482, p = 0.021, \eta^2 = 0.038$ , observed power = 0.64) grade bands (see Table 6). The adjusted mean difference at the elementary grade band resulted in the RGL At-or-Below student performance average being 3.52 percentage points higher than the RGL Above average. The middle school marginal mean difference had the RGL At-or-Below average 5.01 percentage points higher. At both of these grade bands, the students performed better on the assessment items written at-or-below their grade level. As illustrated by the adjusted means in Table 4, there was virtually no difference between RGL At-or-Below averages and

RGL Above averages found at the high school grade band.

Tests of homogeneity were violated at all three grade bands. At the elementary grade band, a significant interaction existed between the RGL groups and TAKS objective covariate ( $F(2, 121) = 6.583, p = 0.002$ ). This did not affect the ANCOVA results; no significant differences in TAKS objectives were found between the RGL groups at the elementary level ( $F(1, 124) = 0.078, p = 0.781$ ). A significant interaction between the RGL groups and student grade level covariate at the middle school indicated homogeneity was violated ( $F(2, 139) = 5.832, p = 0.004$ ). Similarly, because no significant differences in student grade level were found between the RGL groups at the middle school level ( $F(1, 142) = 1.063, p = 0.304$ ), these interactions were not found to affect the ANCOVA results. Homogeneity was violated at the

high school level as well, with a significant interaction between the RGL groups and student grade level covariate ( $F(2, 163) = 29.315, p = 0.000$ ). Like the elementary and middle grades, this interaction did not affect the ANCOVA, having no significant differences in student grade level between the RGL groups at the high school level ( $F(1, 166) = 1.260, p = 0.263$ ).

#### *Overall Assessment Item Analysis*

Controlling for cognitive demand and student grade level, the final level of analysis of covariance determined a significant difference in student performance based on the RGL for all TAKS mathematics assessment items ( $F(1, 434) = 5.926, p = 0.015, \eta^2 = 0.013$ , observed power = 0.68) (see Table 7). The marginal mean difference between the RGL At-or-Below student performance average and RGL Above student performance average was 3.25 percentage points, indicating that students performed better on the RGL At-or-Below questions. Both the cognitive demand ( $F(1, 436) = 1.174, p = 0.279$ ) and student grade level ( $F(1, 392) = 0.313, p = 0.576$ ) covariates were independent from the RGL group variable. However, a significant interaction existed between the RGL group variable and student grade level covariate ( $F(2, 389) = 127.504, p = 0.000$ ). This violation of homogeneity did not, however, have an effect on the ANCOVA, with no significant differences in student grade level found between the RGL groups.

#### **Discussion and Conclusions**

The purpose of this study was to determine if the reading difficulty of mathematics assessment items affected student performance on TAKS items. The claim that student performance would be affected by RGL of mathematics assessment items is supported by the results of this study. Analysis revealed that students in the state of Texas performed significantly lower on mathematics assessment items having RGL measures above their grade level than on items having RGL measures At-or-Below their grade level. This was especially true for elementary and middle school students. More specifically, the seventh grade students in the state of Texas were negatively affected by the RGL of mathematics assessment items.

Despite controlling for several extraneous variables at each level of analysis, the RGL explained very little of the variance as evidenced through the low effect size coefficients. Additionally, the observed power at nearly all levels of analysis yielded results illustrating the limited power of the RGL, except at the 7th grade level, where the observed power coefficient was above 0.8. This evidence of low effect size and

power coefficients either suggests more extraneous variables could be identified or that the limitations of this study had an effect on the results.

In this study, a grade-equivalence measure was obtained for each mathematics assessment item's RGL to determine the item's categorization as being at-or-below versus above the student grade level. This measure constituted the greatest limitation of this study. Like Walker et al. (2008) noted, determining the reading grade level of a small passage of text, like that of a mathematics assessment item, lacks reliability. Lexile (2008) based their calculations on subsets of 125 words within the text sample, and no assessment item in this study had a word count close to 125 words. Therefore the calculations made using the Lexile Analyzer may not have provided the best reading grade-equivalence measure.

Another limitation to this study was the number of assessment items analyzed at each grade level. After items were categorized based on their RGL, nearly all subsets of RGL Above items had less than 20 items. A larger sampling of grade level items would have been ideal. Additionally, the utilization of publicly available data with respect to student performance measures limited the results of this study. Obtaining student level data that can be disaggregated would also be useful because determining how students from different ethnic, socioeconomic, regional, and gender groups could help in identifying how different subgroups of students are affected by the RGL of mathematics assessment items. However, the usage of state level data allowed this study to illustrate effects evident within the overall population of students in Texas.

Overall, this study provides support to the available research indicating the negative effect of a greater reading difficulty on student performance on mathematics assessment items (Bolt & Thurlow, 2007; Powell et al., 2009; Walker et al., 2008). Ideally, assessment items would minimize reading difficulty without jeopardizing mathematical complexity. Therefore, investigating ways of writing mathematics assessment items that require students to read and synthesize text without going beyond the students' reading grade level is imperative. Further empirical research is needed in this area.

Because this study relied on Texas state assessment items in mathematics that are used for determining AYP, implications regarding accountability practices should be considered. If schools and school districts are held accountable for student performance on standardized state mathematics assessments (NCLB, 2002), students (and hence

schools and districts) may be unduly penalized twice, once for low reading performance and once for low mathematics performance resulting from reading difficulties. To provide an accurate assessment of student mathematics performance, student results in mathematics may need to reflect individual student reading difficulties. Adjusted mathematics assessment scores could be created based on individual student reading levels, and these adjusted mathematics achievement results could be used in state accountability measures.

Students are called to acquire mathematical skills that are not only grounded in computation but also in complex problem solving (NCTM, 2000). These complex problem solving skills are assessed with items that unavoidably require a great deal of reading (Bolt & Thurlow, 2007; Walker et al., 2008); mathematics test items will continue to include reading passages in order to accurately present the mathematical situation. Teachers, administrators, test and textbook writers, students, and parents should understand that this dualistic nature of mathematics assessment items is inevitable. However, researchers should continue to investigate ways to minimize the reading difficulty of assessment items without limiting the mathematics content. There may soon be accountability measures that reflect the reading levels of students. But, until that time, teachers must continue to teach both the mathematics content and reading strategies in order for students to perform their best.

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# Mathematics and Martial Arts as Connected Art Forms

Serkan Hekimoglu

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Parallels between martial arts and mathematics are explored. Misguided public perception of both disciplines, students' misconceptions, and the similarities between proofs and katas are among the striking commonalities between martial arts and mathematics. The author also reflects on what he has learned in his martial arts training, and how this wisdom influences his mathematics teaching. As a result of his martial arts training, his awareness of how he teaches mathematics has shifted, and his understanding of his students' struggles has deepened. Finding the balance between theory and practice enhances the process of learning for both students and teacher.

At first glance, it may seem that mathematics and martial arts are conceptually far apart. However, this is not the case. The first thing to understand is that both disciplines are difficult, yet creative, human activities. Martial arts are more than just kicks, punches, and throws; mathematics is not merely a collection of rules, facts, skills, and algorithms. When performed by a skilled practitioner, both are art forms that teach us ways of learning and a framework of thinking that better enables us to use our bodies and minds by maximizing their efficiency. One cannot achieve a high level of skill in mathematics or martial arts by following or executing a collection of rules, facts, and techniques. On the contrary, they are *arts* of exploration, discovery, imagination, and creation. The practitioner enjoys the excitement of searching for new results and techniques, the thrill of discovery, the satisfaction of mastering difficulties, and the pride of achieving mastery.

## Mathematics and Martial Arts

It may come as a surprise that learning martial arts requires as much use of the brain as the body. The word *dojo* means the place of enlightenment. The dojo is a place for facing one's weaknesses and for cultivating a flexible mind and body through hard practice. Both martial arts and mathematics are intersections of art, practical skills, and high ideals that provide a structure to develop an awareness of life through a process of discovery (Devlin, 2000; Funakoshi, 1954; Halmos, 1985; Stewart, 2006). Mathematics and martial arts have a fundamental commonality; in order to master either one, the

practitioner must become skilled at both the mechanical side and the creative, humanistic side. It is possible to perform both mathematics and martial arts using strict rules of deductions and a system of axioms or techniques where all the theorems or moves are then obtained and checked mechanically. However, if you watch a martial arts competition, you will witness a messy struggle punctuated only occasionally with something as beautiful as the *kata*—a synchronized sequence of combative defensive and offensive techniques in a continuous flow. Achieving the beauty and flow of the *kata* takes more than simply following a series of pre-determined steps. This same idea applies to mathematics. The mathematician at work makes “vague guesses, visualizes broad generalizations, and jumps to unwarranted conclusions. He arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof” (Halmos, 1968, p.380).

While solidly built on ancient traditions, countless practitioners have further developed both disciplines by devising and polishing techniques, concepts, and ideas. With every generation, martial arts and mathematics evolve through accumulated knowledge, techniques, concepts, perceptions, and experiences built upon by past practitioners. The concepts and techniques continue to change over time. Not only are new concepts and techniques developed, but at the same time old concepts and techniques are reworked, modified, and redefined (Bolelli, 2008; Davis & Hersh, 1983; Halmos, 1968). The practitioner, therefore, can only gain a proper understanding of martial arts or mathematics through constant practice or study that is not limited to a technical perspective, but also includes a historical and cultural perspective. Learning mathematics and martial arts will have a profound effect on the student since the community plays a powerful role in shaping both the works and lives of

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their practitioners (Bolelli, 2008; Ernest, 1998; Gonzalez, 1989).

One can perceive martial arts and mathematics as an amazing range of mountains without a single peak. We might then describe practicing these arts as climbing an endless mountain, with different routes to the same elevation. As the climber scales the mountain, the view below changes. Able to see more of the surroundings, the climber's sense of where he is and what really exists in the world changes. Continuing to practice martial arts or mathematics, he will find himself able to move his mind in new ways and gradually discover new strengths that can expand his mental horizons. A mountain climber can always try to reach higher elevations or can choose to be content with reaching a certain plateau, even though there are higher peaks in the range. As one climbs higher, the view and the connections between points become more interesting and more intriguing. Martial arts and mathematics also offer many challenges, both external and internal. The difficulty of certain movements, the complexity of the concepts, exhaustion resulting from rigorous practice and study, and the pain of sore muscles or headache can produce a great deal of frustration and discouragement. The journey for each individual is unique. A master or a teacher can illuminate principles behind techniques and concepts but one must discover the truth for oneself.

Mathematics and martial arts are pilgrimages of self-improvement; driven by human desires to find perfection and purity in the human mind and body by uncovering the hidden simplicity and complexity that coexist in the world (Halmos, 1985; Konzak & Bourdeau, 1984). In both disciplines, the knowledge is not so much something that one possesses, but rather is a process of self-discovery. One constructs mathematical ideas or martial arts techniques internally, as a way of dealing with a perceived problem. Therefore, the nature of the objective governs the selection and the use of tools, whether they are legs, arms, concepts, algorithms, or techniques. On the journey toward mastery in mathematics or martial arts, the practitioner learns to combine ideas or techniques through experience, hard work and recognition of what is important. Eventually, the practitioner may feel as though he is no longer simply using tools and concepts as presented to him, instead using their combinations to create something new. Depending on the amount of commitment and energy the practitioner has put into training and studying, there are feelings of hard-won sense of accomplishment, satisfaction, and self-improvement.

### Misguided Public Perception of Both Disciplines

Martial arts and mathematics both suffer from a misunderstanding by the general public. Both disciplines have received an unfortunate public image, which is quite different from the perspective of the practitioner (Funakoshi, 1954; Stewart, 2006). While the general public usually considers having a PhD in mathematics or a black belt in martial arts a mark of expertise, the practitioners of both perceive these achievements simply as mere demonstrations of a committed student (Halmos, 1985; Layton, 1988; Stewart, 2006). Furthermore, in addition to being confused regarding the goals of martial arts practice or learning mathematics, the general public is equally clueless as to the true benefits of martial arts and mathematics. Many people view mathematics as an abstract, non-creative, body of knowledge that is to be memorized and applied in a mechanical way (Devlin, 2000; Schoenfeld, 1989). On the contrary, mathematics is a science of patterns which demands creativity. Mathematics requires the use of a vivid imagination, a sense of scientific beauty, and the ability to reason in selecting ideas and concepts (Halmos, 1968). In a similar vein, the true benefit of martial arts does not lie in its sporting value or as a means of fighting, but in the opportunity it provides for becoming a stronger, more complete individual.

Current movies provide a much-distorted picture of what mathematics and martial arts really are, as the philosophy and the subtle beauty of the arts do not come across well on the screen. Movies about mathematics (e.g., *Pi* and *Proof*) frequently provide a negative image of mathematicians by portraying them as loner sociopathic savants. At best, movies may depict a mathematician as an absent-minded nerd engrossed in scribbles and equations, or as a kind of human calculator who can perform complicated mental calculations with amazing speed and accuracy (Burton, 1989; Furinghetti, 1993; Hekimoglu & Kittrell, 2010; Lim, 1999; Mendick, 2002; Picker & Berry, 2000). The negative impact of these movies is their unrealistic representation of the mathematics problem-solving process. For instance, the crime drama *Numb3rs* depicts the main character solving problems in less than a day. However, in reality, a cadre of mathematicians might take months to solve such problems.

In striking comparison, the violent martial arts movies contribute to the corruption of the discipline by portraying the stereotypical image of a martial artist as a bare-handed, acrobatic, *Marlboro Man* who screams with flying exotic high kicks (Layton, 1988; Reiter,

1975). The spectacular and flashy movements that require excellent athletic abilities are highly unrealistic with regard to fighting. Many movies provide a romantic illusion of fighting, along with a fantasy of what it takes to master in martial arts. The portrayal of Daniel Larusso in *The Karate Kid* provides an example of a martial student who trains with a “master” for a short period of time and rapidly becomes proficient in karate-dō (Weintraub & Avildsen, 1984). Even worse, some recent reality shows present an unattractive image of martial artists by portraying them as mindless jocks or buffoons, e.g., *The Ultimate Fighter Reality Show*. Both learning mathematics and training in martial arts are vastly complex endeavors that require intense concentration in order to succeed (Barnfield, 2003; Brown, 2003; Halmos, 1968; Hardman, 1954). The transition from uninformed enthusiast to committed student is a gradual one because it takes time to develop competence by going through a slow and constant contemplative process of change and improvement. Gradually, with practice, reflection, and experience gained through handling different opponents or solving problems, one begins to understand what mathematics or martial arts are really about.

### **Learning Mathematics vs. Practicing Martial Arts**

Ideas and visions form the basis for the practice of both mathematics and martial arts. The process of learning in these disciplines is a series of realizations or awakenings; the harder one studies, the more fascinating the arts become (Bolelli, 2008; Brown, 2003; Gonzalez, 1989; Halmos, 1985; Stewart 2006). The practitioners need to make a healthy obsession of technical details. It is one thing to understand the techniques and concepts, but it is quite another to know them intuitively. In mathematics and martial arts, practitioners must repeat certain movements, techniques, exercises, and algorithms many times so that they can become part of their natural reflexes or thought processes. Practicing a technique or algorithm repeatedly not only makes one more proficient, it also trains and develops his or her neuromuscular or cognitive system to act, respond, or think in accordance with the technique or concept. The outcome of a successful learning experience is either an *assimilation*, the integration of new understanding into the existing neuromuscular or cognitive structures, or an *accommodation*, a reorganization of the existing neuromuscular or cognitive structures in order to allow one to develop these structures on higher levels of organization (Piaget, 1985; Steffe & Wiegel, 1996; von Glasersfeld, 1995). Through a series of assimilations

and accommodations, the connections become more interesting and more nuanced. The student’s understanding becomes more refined as he or she begins to relate to more subtle dimensions of techniques and algorithms by examining why they work and what constitutes the elements of their effectiveness.

Progressive skills and knowledge development are keys to long-term progress in both mathematics and martial arts since everything that one learns is merely a preliminary foundation for the next level. Learning in martial arts and mathematics is like building a house. A solid foundation is required so that the structural integrity of the house remains intact. Similarly, practitioners need to take the time to build a solid foundation of basic skills and concepts, and constantly refine and add to this base so that they can expand their knowledge. There are neither concepts in mathematics, nor skills in martial arts, that can exist without a foundation. Therefore, failing to develop a proper understanding of fundamental concepts or skills prevents the student from improving and refining his skill level. When one learns a new martial arts technique or a mathematics concept, he or she must incorporate the elementary principles they already know with the new knowledge in order to broaden its scope and applications. When it is difficult to grasp a new step or concept, a student needs to break it down by isolating the appropriate relationships and properties, and then practice or study them separately through continuous self-reflection (Gonzalez, 1989; Hardman, 1954; Skemp, 1971; VonGlasersfeld, 1995). The learning process starts with the introduction of basic concepts or techniques; the instructor then gradually increases the complexity and difficulty of the material as a student advances. In martial arts, the student starts with learning basic punches, kicks, blocks, and stances. Once comfortable with the basics, the student learns how to put them together in kata and fighting practices. Similarly in mathematics, as the student’s knowledge grows, new ideas and concepts are introduced that build upon the previous ones.

To become experts in both disciplines, students must not only acquire facts, but also organize their knowledge to facilitate the application to diverse situations. It is this understanding that makes one a mathematical expert or a formidable fighter and enables him to use the knowledge or techniques creatively, flexibly, and fluently, in different settings or problems. The learning process requires the ability to shift attention from the objects or techniques to the structure of their properties and relationships. Later,

the student needs to compose parts in such a way that they form a coherent whole. For example, one cannot look at proofs and katas as if they were arbitrary collections of steps or techniques. There is a need to understand each step or technique and how each is related to previous and proceeding ones in the proof or kata. The student should be able to see the proof or kata as a single object by putting the steps back together into one complete object or technique.

### **Proofs and Katas**

Martial arts and mathematics instructors know well the problems that students have appreciating the need to practice katas or complete proofs. We often get frustrated when we hear students saying that practicing katas is boring or that practicing katas does not help them learn how to defend themselves. Neither do we like to hear students questioning the importance of proving mathematical theorems. What functions do katas and proofs have within martial arts and mathematics and what makes the practice of them a meaningful activity? First, katas and proofs provide the glue that holds martial arts and mathematics together; they serve as a means of systematization in both disciplines. In mathematics, proofs help us to systematize various known mathematical results into a deductive system of axioms, definitions, and theorems. In martial arts, a kata unifies techniques by integrating unrelated kicks, punches, and blocks, leading to an aesthetic and efficient presentation of movements. Another function of proofs and katas are that they are forms of discourse. Both serve as a medium for communication and validation of traditions among people who share similar backgrounds (Bolelli, 2008; Hopkins, 2004; Davis, 1976; Funakoshi, 1954; Gale, 1990; Gonobolin, 1954; Hanna, 1989; Tall, 1989).

Katas and proofs also serve as the standard measure of the technical basis of competence. A student's understanding of martial arts or mathematics can be seen in his performance of the kata or in providing proof of a concept. Additionally, proofs and katas can serve as a challenge. Mathematicians find the process of doing mathematical proofs appealing because they test their knowledge and creativity. To martial artists, katas provide a physical challenge that they find as appealing as the mental challenge of a mathematical proof (Campell, 2005; Manin, 1981; Renz, 1981). Lastly, proofs and katas are teaching and learning tools. Both help to acculturate students in the discipline since they embody lessons learned by past masters (Campell, 2005; Hopkins, 2004; Wilder, 1994). The execution of katas or proofs will provide a student with some of the most effective fighting or

mathematical techniques ever developed. The techniques in katas or ideas in proofs can also be a springboard for further techniques or concepts not found in the particular kata or proof under study. They serve as important tools for clarification, validation, and deeper understanding (Bolelli, 2008; Campell, 2005; Fischbein, 1982; Funakoshi, 1954; Gonobolin, 1954; Hopkins, 2004; Tall, 1989; Van Asch, 1993; Van Dormolen, 1977; Volmink, 1990).

### **Achieving Mastery**

Mastery in math and martial arts does not just happen, one achieves mastery over time. Achieving mastery is a slow, gradual, and often frustrating process (Brown, 2003; Hobart, 2006; Stewart, 2006). Thus, patience is an essential quality of both martial artists and mathematicians. Discipline is crucial since the improvement is a gradual, day-by-day process. One can only achieve genuine success by making full use of those valuable experiences sometimes referred to as failures. There is no shame in being knocked down by an opponent or being unable to solve a problem. Once you have learned how to turn pain and frustration into self-knowledge and personal growth, the challenges focus more on what is being learned and how it can be developed more fully. Only those interested in the higher ideal will find martial arts interesting enough to persevere through the rigors it entails (Halmos, 1985; Bolelli, 2008). Those who do will find that the harder they train the more fascinating the art becomes. While martial artists pay for their expertise with sweat, bruises, and blood, mathematicians pay the price with many sleepless nights and headaches. The more time one spends doing mathematics or the harder he trains in martial arts, the more one begins to appreciate the true depth and beauty of each discipline. This new appreciation does not mean that his previous understanding was wrong; it simply means that he has moved on to a higher level (Hardman, 1954; Richman & Rehberg, 1986).

Struggles are also a normal part of both mathematics and martial arts training processes. Without perseverance, there is little chance of ever pushing through the hard times. Breakthroughs result from sustained effort. In both disciplines, the way to true understanding must lead through personal experience and suffering. Even though there are natural stages in the development of a martial artist or a mathematician, it takes effort to move from one to the next. Only those who constantly renew their commitment to study and train with interest and enthusiasm will attain the highest level of achievement. When you hit a wall in your learning, the key to



overcoming the barrier is to immerse yourself completely in the problem or technique. As grandmaster Gichin Funakoshi (1954) expressed, “you must be deadly serious in training...I do not mean that you should be reasonably diligent or moderately in earnest” (p. 105). Paul Halmos (1985), one of the leading mathematicians of the twentieth century expressed similar ideas by saying that learning mathematics requires complete focus and loyalty: “To be a mathematician, you must love mathematics more than anything else, more than family, more than religion, more than any other interest” (p. 400).

### **Overcoming Disillusionment and Attrition**

The students of typical martial arts dojos or mathematics classrooms are extremely heterogeneous. Each student brings a unique set of strengths, weaknesses, interests, ambitions, responsibilities, levels of motivation, and approaches to studying or training. Differing physical capacities or mathematical knowledge, emotional maturity, and psychological factors create varying dynamics for each student. This means that instructors in both disciplines must become comfortable with the idea of individualizing instruction for their students. Teachers should adjust the vigor and degree of difficulty in sparring and the difficulty level of mathematics problems to the student’s current developmental level (Piaget, 1985; Vygotsky, 1978). The instructor should adjust and enrich the curricula through differentiations in pace and depth, as well as making changes in their teaching style to match the way students learn. In both disciplines, the instructor’s judgment is extremely important in knowing when to press onward with intensive training to stimulate learning and when to stop in order to avoid student injury or discouragement.

Not surprisingly, attrition remains a significant problem in both endeavors. The slow process of growth is often unbearable to many students who have come to expect instant gratification. Students often have unrealistic expectations of what they can achieve with martial arts or mathematics and how quickly they will be *finished*. Many students want to get their black belts or get an A in their mathematics class to gain a sense of self-confidence and success (Middleton, 1995; Layton, 1988; Reid, Wood, Smith, & Petozc, 2005). Students should practice martial arts and mathematics for their own sake. One must be willing to spend time outside of regular practice or class time to fully internalize the techniques, concepts, algorithms, or movements. In martial arts, the realization that mastery can be achieved from endless training has given way to the more popular fantasy of an easily won black belt

status after a few months’ work (Richman & Rehberg, 1986). Likewise, in undergraduate mathematics classes, students generally receive a rude awakening after the first exam when they realize that they cannot begin studying one or two nights before the test and expect to do well on the examinations. Frequently, students will become disillusioned with the amount of hard work required to excel, and so a large percentage of students of both disciplines will drop out or fail (Brown, 2003; Grady, 2000; Hobart, 2006; Jackson & Leffingwell, 1999).

The truth is that there are no shortcuts or magic formulas for learning mathematics or martial arts. The key to success in both disciplines is simply to become personally accountable for what you learn or do not learn, and to practice or study as often and as hard as you can. The skills that look so easy when performed by a master martial artist or a mathematician are not the result of the martial artist’s unique body or the mathematician’s unique mind; their performances are the result of long, hard, and dedicated practices.

### **Real-World Applications**

One should learn the real-world applications of the techniques of martial arts and the concepts of mathematics in context. A single movement or concept will have several different applications, and the ideas and techniques can be adapted to achieve various goals. Bridging the gap between practice and real-world applications will help students to develop a proper understanding of what martial arts or mathematics is and how it relates to the rest of the world. The dilemma is the trade-off between content and real-world problems in mathematics classes or forms and fighting in martial arts classes. To learn the fighting lessons of martial arts, a student must experience a physical encounter through an unchoreographed exchange of techniques (Alsina, 2001; Grady, 2000; Kloosterman, 1996; Olson, 2003; Stewart, 2006). Similarly, students need to see the application of mathematics in different academic disciplines, where extraneous variables complicate problems or standard algorithms are insufficient.

Teachers must inject realism into a student’s training, because actual violence differs greatly from choreographed training in the dojo, and real problem solving processes differ significantly from the polished proofs in mathematical journals. In both martial arts and mathematics training, it is the instructor’s job to challenge students to seek new levels of excellence. In martial arts, the instructor should help students to avoid developing false confidence while working with smaller training partners. Until a student is certain

techniques work against a larger person, the student has not learned self-defense. In mathematics classes, students sometimes shy away from working with complex problems and the instructor must challenge students to reach beyond their comfort level and increase their knowledge base. Without knowing the applications of the art, studying mathematics becomes merely a mental exercise, and training in martial arts is no more than exercise for the body.

### **Teaching Mathematics Based on Martial Arts Principles: A Personal Story**

As a longtime martial arts practitioner, I have discovered that I can apply the principles of martial arts to the teaching of mathematics. When free sparring, the goal is to learn from an opponent and to remain deeply attentive. It is imperative to make no assumptions regarding one's own actions or those of the opponent. One must try to develop the correct understanding of the opponent's movements and the proper techniques for responding to them. Translating this basic principle to my life as a mathematics instructor, I strive to be fully present and connected in the classroom. As with martial arts, each teaching moment requires constantly adjusting to the needs of the student. At the beginning of my teaching years, my focus was on the mathematics, not the students. I used to think the students were in the classroom for the mathematics, not that the mathematics was there for the students. If I truly want to motivate my students, then I must find a way to reach their interests. It took me a while to realize that my students do not really care how much mathematics I know. Instead, what they need to see is how much I care about teaching them mathematics. A good instructor must act in harmony with the students, and remember to be the teacher of the students you *actually* have, not the students that you might *wish* to have.

Through my martial arts training, I have learned that it is necessary to develop a sense of self-esteem and mutual respect between the instructor and students in the dojo. A good martial arts instructor never tries to impress students with his own skills and knowledge. His motto is not let me show you what I can do, but rather let's see what we can do together. When I apply this philosophy to the mathematics classes I teach, I know that my mathematical knowledge may give me power, but it is my character that earns the respect of my students. I strive to model excellence for my students. This helps to build trust and respect, and will hopefully encourage them to raise their level of performance. As an instructor, it is essential to be knowledgeable, challenging, organized, clear, and fair.

But these characteristics matter little without the desire to encourage students' learning (Jackson & Leffingwell, 1999; Hekimoglu & Kittrell, 2010; Schon, 1987). It is just as important to be committed, enthusiastic, and genuinely warm to motivate students to give me their best, and to encourage them to strive for excellence in everything they do. As my martial arts instructors did for me, my job as a mathematics instructor is to create a stimulating classroom environment that inspires effort and achievement.

Another lesson from my free sparring sessions that I have integrated into my teaching of mathematics is the adoption of basic karate principles of *ikken hissatsu* (finish with one blow). In free sparring, one tries to finish an opponent with one strike without using fancy or complicated maneuvers. In my classes, I create lesson plans based on this principle. I try to pare away anything convoluted and confusing by presenting the concept and ideas in a clear and logically progressive manner. Furthermore, free sparring has taught me to always consider the possibility that I may be unable to conquer my opponent. Likewise, a mathematics instructor should consider the possibility that they may be unable to reach their students with their primary teaching method. When you teach or initiate an attack in free sparring, you should always try to gauge the reaction of your students or opponents before you proceed. The experienced martial artist or mathematics instructor guides his actions by his opponents' or students' reactions.

One of the most important things that I have learned in martial arts training and have integrated into my mathematics teaching is to make my class a place where students can confront their anxieties and fears. To become a good fighter, every martial arts student must learn to face fear. If you attack with the fear of being injured, your attack will not be fully committed and the probability of being injured increases. For students of mathematics, the real enemies are the doubt, confusion, and fear within the students themselves. A student must learn to overcome the frustration, discouragement, and even depression that can result from failure to make satisfactory progress. The presence of fear and anxiety will inhibit the progression of learning (Garofalo, 1989; Hackett & Betz, 1989; Hall & Ponton, 2005; McLeod, 1994).

The essence of teaching mathematics lies in leading students to believe that they can learn mathematics (Crawford, Gordon, Nicholas, and Prosser, 1994; Kloosterman, 1996; McLeod, 1994). They must be able to visualize success, instead of focusing on the chance of failure. I always try to create

an open and positive environment where setbacks, mistakes, errors, and failures are permissible. In this way, students can explore their potential without fear of judgment or criticism. I have also learned that testing in both martial arts and mathematics is simply an opportunity to reflect on the student's progress and allow them to acknowledge their strengths, weaknesses, and discover areas for self-improvement. As an instructor, I must help my students to realize their own ability to go beyond their self-imposed limitations.

Training in traditional martial arts is one of the most valuable pursuits I know. The more I began integrating martial arts principles into my teaching, not only did my outward success grow, but more importantly, my sense of being true to myself brought me a deeper satisfaction. Learning to teach mathematics and martial arts training are both ongoing journeys, with each new experience leading to a new challenge. The real secret to becoming an expert in both martial arts and mathematics instruction is realizing that the learning is a process of self-discovery (Bolelli, 2008; Schon, 1987). By striving to perfect one's self-ability and understanding of the abilities of others, wisdom in the discipline develops. The principles of martial arts have become a medium that have given me the means to expand my potential and to enhance the experience of my students.

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# Expert Mathematicians' Approach to Understanding Definitions

Revathy Parameswaran

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In this article I report on a study of the cognitive tools that research mathematicians employ when developing deep understandings of abstract mathematical definitions. I arrived at several conclusions about this process: Examples play a predominant role in understanding definitions. Equivalent reformulations of definitions enrich understanding. Evoked conflicts and their resolutions result in improved understanding. The primary role of definitions in mathematics is in proving theorems. And there are several stages in developing understandings of mathematical definitions (Manin, 2007; Tall & Vinner, 1981; Thurston, 1994). I also include some suggestions for pedagogy that are found in the data.

Definitions play a pivotal role in mathematics. Research on students' understandings of mathematical definitions reveals that learners encounter different types of obstacles. According to Vinner (1991), serious difficulties in comprehending definitions can be attributed to the dichotomy that exists between the structure of mathematics as conceived by professional mathematicians and the cognitive processes involved in concept acquisition by learners. Hence, it is instructive, based on the resulting pedagogical implications, for both mathematics teachers and educational researchers, to understand how professional mathematicians view mathematical definitions and what cognitive processes they employ when they attempt to understand definitions created by their peers.

With this goal in mind, I conducted a qualitative research study with 12 professional mathematicians. The mathematicians in this study have made lasting contributions to mathematics, and many have won several international awards and honors. For the purposes of this article, I define a professional mathematician as a mathematician who is actively involved in mathematical research. An expert mathematician has additionally been internationally recognized by peers based on his or her profound findings. In this research study, I only solicited participants who are expert mathematicians in their respective fields. The participants' responses provided insight into their world of mathematics, particularly

with respect to definitions. At times, their responses went beyond their understandings of mathematical definitions, allowing important themes to emerge, such as the role of examples, conflicts, models, abstraction, intuition, and generalization. In order to adequately represent the emergent themes from participants' responses, I have also relied upon writings on the nature of mathematics alluded to by some of our respondents. In the following sections I present the views expressed by the mathematicians on each of the aforementioned themes.

## Theoretical Perspective and Literature

The theoretical framework of Pirie and Kieren (1994) is relevant to this study, for it describes the development of understanding within the learner's mind when a mathematical concept is learned. This theory describes the dynamic growth of understanding over a period of time. The essence of their theory is that understanding is not always a linear, continuous process; learners often revert back to their previous ways of thinking, only to emerge forward with more sophisticated and deeper understanding.

Pirie and Kieren propose an *onion-layer* model to depict eight different levels of understanding within the learner (see Figure 1). The innermost level of the model is referred to as *primitive knowing*, for this level describes the process of initial attempts to understand a new concept (such as functions) through actions involving the concept (adding or composing functions, evaluating a function at a point, etc.) or representations of the concept (such as the graph of a function). In the next level, the learner develops images out of these effective actions. This level is called *image making*. Continuing outward, the next level is called *image having*. At this level the learner is able to refine and manipulate the image (such as the image of a conic

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associated to a quadratic function) without having to work out particular examples, and, hence, this level represents the learner's first level of abstraction.

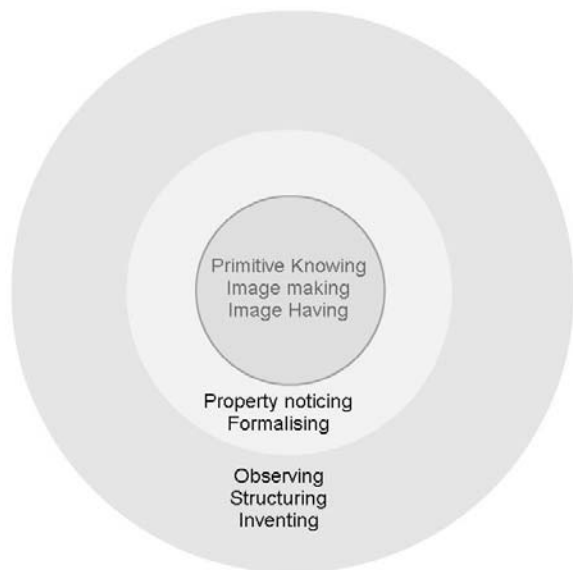


Figure 1. Pirie-Kieren's model

The next level is called *property noticing*, in which the learner is able to examine these images for properties, distinctions and so on. At this level the learner may notice, for example, that certain conic sections such as circles and ellipses are bounded, whereas hyperbola and parabola are unbounded. The model's next level is *formalizing*: The learner thinks consciously about the noticed properties and is able to generalize by abstracting the important features of the mathematical concept. At this level, the mathematical concepts become defined for the learner and begin to exist as an independent entity.

At the level titled *observing*, the learner tries to achieve consistency in his or her thought processes by trying to accommodate existing knowledge structures to fit with the newly acquired knowledge. For example, the learner may base his or her properties of functions on a modification of the properties of numbers, noticing that functions, similar to numbers, can be added and multiplied, and yet, unlike numbers, functions can be composed. The level called *structuring* takes place when the learner is able to place his or her thought processes into an axiomatic structure. At the outermost level, appropriately named *inventing*, the learner is able to freely create new mathematical structures with the previous knowledge structures acting as the initiating ground. At this final

level, in which the highest level of recursion occurs, the learner begins to function independently. It is important to note that the levels do not correspond to levels in mathematics, but, rather, levels in understanding. Thus this theory reflects understanding as a personal knowledge construction process.

Vinner and Tall (1981) provide a framework for understanding how one understands and uses a mathematical definition. This framework is foundational to this study in that it explains the dynamic interaction between the concept image and concept definition in addition to being influential in subsequent research on the role of concept-images in understanding formal mathematical definitions. According to Tall (1980), each mathematical concept is associated with both a concept definition and a concept image.

Concept image is regarded as the cognitive structure consisting of the mental picture and the properties and processes associated with the concept. Depending on the context, different parts of the concept image may be activated. At any given time the portion of the concept image that is activated is called evoked concept image. Quite distinct from the complex structure of the concept image is the concept definition which is the form of words used to describe the concept. (pp. 171–2)

The mathematical definition could be formal and given to the individual as a part of a formal theory or it may be a personal definition invented by an individual describing his or her concept image. A *potential conflict factor* (Tall, 1980) describes any aspect of the concept image that may conflict with any part or resulting implication of the particular concept definition. Factors in different formal theories can give rise to such a conflict. A cognitive conflict is created when two mutually conflicting factors are evoked simultaneously in the mind of an individual. The potential conflict may not become a cognitive conflict if the implications of the concept definition do not become a part of the individual's concept image. The lack of coordination between the concept image developed by an individual and the implication of the concept definition can lead to obstacles in learning. This has been corroborated by the work of several researchers including Cornu (1991) and Edwards and Ward (2004).

The influence of concept images on the understanding of mathematical concepts has been extensively studied. Nardi (1996) observes that novices are often obstructed by their previous unstable knowledge. Alcock and Simpson's (2002) research

observes that students generally do not consult definitions to resolve conflicts because they do not understand either the relevance or importance of definitions. When learning certain concepts, learners also face difficulties due to the concept's intrinsic complexity. Bachelard (1938) classifies learning obstacles into several types according to their source: use of particular language, association of inappropriate images, or effect of a previous piece of knowledge that was originally useful but which becomes false in the present context. Thus research reveals that learners, in general, find it difficult to comprehend and use definitions.

### Research Methodology

This research began with a pilot study exploring how expert mathematicians approach and understand mathematical definitions. I conducted in-person interviews with two expert mathematicians concerning the cognitive processes involved in their understanding of mathematical concepts and definitions. During the analysis of the pilot study data, the principal themes of the role of examples, conflict resolution, and reformulations and generalizations emerged. The emergent themes provided focus for subsequent research and informed the development and preparation of a questionnaire.

#### *Participants*

Twelve expert mathematicians participated in this research study. Although the research interests of the participating mathematicians varied, all participants study and research pure mathematics. Most of them have also taught mathematics for many years.

#### *Data Collection and Methods*

Because the participating mathematicians live in different parts of the world, only 5 participants took part in personal interviews. All participants responded to the questionnaire. Although no new questions were put forth during the interviews, this approach provided more time to reflect on the questions given.

The intent of the questionnaire (see Appendix) was to understand how expert mathematicians comprehend definitions. Guided by the emergent themes of the pilot study, the questions focused on the roles of imagery, examples, conflicts, as well as the domain to which a given definition belongs.

#### *Analysis*

Analysis of the participant responses used grounded theory as defined by Glaser and Strauss (1967). In particular, my analysis followed Glaser and

Strauss's method of open coding of data. Using this approach, I constructed meaningful patterns within participant descriptions by looking for structure in the data. Themes emerged as a result of analyzing the responses for commonalities and differences.

### Results

#### *Processes Used by Mathematicians in Understanding Definitions*

*The role of examples.* I define *examples* as instantiations of a concept. Every participating mathematician discussed the importance of examples in developing their understanding of mathematical definitions. This research finding supports the contention that having a sufficient number of suitable examples is closely linked with the understanding of the mathematical object. The *image forming* and *image having* levels of Pirie and Kieren's (1994) onion-layer model seem to be related to identifying examples and non-examples associated with a particular definition. By making sense of theorems that rely on a given definition, these images are further strengthened. For the participants, intuition, in any particular area, is also built by having a rich source of examples.

The following responses highlight the different ways expert mathematicians use examples to develop their understanding of mathematical definitions.

**Response 1:** To comprehend a definition means it usually includes examples, simple counter-examples, and theorems using the definition.

**Response 2:** I try to see how the definition will exclude/include examples that I already know. For example, if the definition is about groups then I would try to see whether it clearly divides the groups that I know into those that fit and those that do not.

**Response 3:** In my area of my expertise, it is less of a challenge for me to comprehend a definition, as I may have already built up an intuition in that area, and examples are already swirling in my head upon reading a definition. In an area less familiar, I cannot feel like I understand a definition (or a theorem for that matter) until I have checked myself that it is not an empty definition (or theorem). I immediately try to think of easy examples, try to make a picture or connect it with some definition I already know and see how it differs in comparison.

Additionally, some of the participants used examples specifically as scaffolds to build their

understanding. This use of scaffolding is seen in the responses that follow.

**Response 4:** Examples are scaffoldings as one tries to build one's understanding of definitions. They are the steps to attain higher and higher levels of understanding. They are also the pillars on which the definitions rest. The more examples one has, the closer one is to understanding the mathematical object. If you want to go from A to B, there may be several ways. Different examples provide the different routes. In fact, examples give approximate shape of the object that is defined. So, complete understanding is impossible. As you make interconnections you get a finer and finer picture of the object.

**Response 5:** For me, any definition is associated with examples. Definitions cannot stand abstractly without examples. For me, knowledge can be thought of as subdivided into islands. It is examples that connect them. So a definition is a collection of all the examples that will conform to the definition. For me, in some sense, all the examples that satisfy a definition is like a kernel of the definition. The construction of examples and analogies are so important in understanding definitions in all the subjects you mention.

*The role of conflicts.* Based on the responses of the participants, when one receives external stimuli in the form of a new mathematical definition, one tries to incorporate this new knowledge to one's existing knowledge structure. This can be influenced by distorted images, resulting in a conflict. According to our participating mathematicians, when the same mathematical object is viewed from a different perspective, conflicts may be resolved. Additionally, there are times when the usual meaning associated with a word used in a new definition also can be a cause of conflict, as the first response below indicates.

**Response 1:** I guess it might also sometimes be required to ignore the usual meaning of the word being defined, but this always came naturally to me.

**Response 2:** I recall conflicts occurring when I convinced myself that the definition meant a certain something, and then later on I encountered a counterexample! I had to re-evaluate my understanding and re-understand it correctly so that the paradox could be resolved.

*Methods of understanding.* The participants' responses suggest that encountering alternate definitions can increase understanding. Similarly, comparing and contrasting the new definition with an

already known definition is a tool used for understanding (see Response 3). Additionally, participants commented on the utility of proving theorems based on a given definition to develop their understanding of that definition. Participants detailed how self-inquiry, or posing questions to oneself, increased understanding.

**Response 1:** I had once a professor of geometry who told us his main aim was that we understood ...[I don't remember if it was "cube", "tetrahedron", or "projective plane"] as well as we understand what "chair" means. We can recognize a chair, sit on one, or in case of need stand on one to reach a high shelf.

That is what, for me, it is to comprehend a definition. It usually includes examples, simple counterexamples, and theorems using the definition. It happens that "definition" and "theorem" can be interchanged, and sometimes this makes a better understanding.

**Response 2:** When one meets an elegant result in an area, one usually marvels at the proof. If I would like to understand this at a deeper level, I usually try to formulate a question in which these notions would intervene and this way, I learn to appreciate those concepts as well as the techniques.

If the definition is equivalent, I note it in the back of my mind that this is an equivalent formulation. At times, the alternate way of looking at things proves useful in solving problems.

**Response 3:** I try to make a picture or connect it with some definition I already know and see how it differs in comparison.

Response 4, below, sums up many of the methods for understanding that were discussed by the participating mathematicians. It is also consistent with the framework developed by Vinner and Tall (1981): The participant hypothesizes that when a learner encounters a new formal mathematical definition, he or she develops a concept image associated with the definition. When this concept image becomes a useful tool, it may replace the definition, or, rather, it may become the concept definition.

**Response 4:** Understanding (comprehending) a mathematical definition is a process which is in principle open-ended: you can never tell that you understand something completely. It can be conceived as consisting of several stages.

Stage 1. Understanding the language in which the definition is stated. In mathematics, I will take for granted that this means the language of more or less formalized set theory, which is expressed in metalanguage based on some natural dialect: English, Russian ... All other choices lack



universality, conciseness, common acceptability, etc., of Set Theory. However, they might be unavoidable at earlier stages of studying mathematics, e.g., if one is taught Euclidean geometry a la Euclid.

Stage 2. (a) Understanding of the Definition itself as a syntactically correct and meaningful expression of the language of Set Theory. (b) Forming imprecise but intuitively helpful “semantic cloud” of the definition. Let’s take as a representative example the definition of a group. Its “semantic cloud” consists in various ideas about symmetry: symmetry of “things”, symmetry of patterns, symmetry of physical laws ...

Stage 3. Trying to compile a list of “examples”: concrete objects satisfying conditions of the Definition. “Small” objects: groups with one, two, three elements. “Big objects”: integers, rationals, matrices ... Can one classify small objects? Describe explicitly groups of 1,2,3,4,5,6 elements up to isomorphism? Here one more definition crops up: that of isomorphism to which the same (up to now, three stage process must be applied).

Stage 4. Studying how the Definition works in various theorems about groups, and in various theories where groups are not the central, but an important part of the picture. Where and how we use associativity, existence of identity, existence of inverse element? When there is a chance that all these conditions for a composition law will be satisfied, and when not? Does a given theorem remain true if one omits existence of inverse element in the definition? What kind of “group-like objects” [do] we get then?

All stages, but especially Stage 4, is in principle open-ended. It might involve returning to Stage 3, posing and solving classification problems, sometimes marveling at their complexity (classification of simple finite groups). It enriches our grasp of semantics of the basic language and in this way helps to understand further definitions.

The various stages involved in understanding mathematical definitions described in this response also resonate with the levels of understanding given by Pirie and Kieren (1994). Forming a semantic cloud with a good supply of examples related to small and big objects corresponds to primitive knowing, image making and image having levels. Identifying patterns, asking questions and proving theorems related to the definition is similar to the level of property noticing, formalizing and observing. It becomes apparent through the Pirie-Kieren model and the participating mathematician’s description that the levels are nested within each other and not necessarily linear.

The responses in this section can also be linked to Lakoff and Núñez’s (2000) work, which demonstrates how mathematicians conceptualize mathematical objects through conceptual mapping. Just as in mathematics, where a mapping (or a function) has a domain and a range or target, the domain of the conceptual mapping consists of examples and the target is the algebraic structure that underlies the set of examples. It is through such metaphorical mappings that mathematicians assign an algebraic essence to an arithmetic structure. They claim that mathematicians tend to think of the algebraic structure as being present in the arithmetic structure. Calling this a metaphorical idea, Lakoff and Núñez propose that this metaphorical idea helps mathematicians to see, for example, the same mathematical structure in addition modulo 3 and rotational symmetries of an equilateral triangle.

*Nature and role of mathematical definitions.* When mathematicians communicate their results in a formal way (e.g., a journal article), a need arises to introduce a collection of associated definitions in order to facilitate “chunking” and help avoid repetition. According to David Mumford (2001), when one encounters many complex examples, isolating part of their shared structure is the best approach. This is what generating models is all about, and, as he points out, pure mathematics is full of models. This method of generating models is referred to by Mumford as the “bottom-up view”. The opposing view is the “top-down view” where different branches of mathematics grow out of one *true* axiomatization of the subject. This “top-down view” is contrary to the ideas of generating mathematical models (pp. 4–5). These two viewpoints also reflect the two different approaches used by mathematicians in solving problems and developing theories. Some mathematicians work from concrete examples, abstract their essence, and generalize theorems. Some mathematicians, by intuition and by virtue of their mathematical experience, conjecture some mathematical statements to be true, verify it by examples and then prove them formally. Mumford points out that, according to Sir Michael Atiyah, the most significant aspects of a new idea are often not contained in the deepest or most general theorem resulting from the idea, but they are often embodied in the simplest examples, the simplest definitions and their immediate consequences (p. 3).

The following participant responses depict additional emergent themes on the nature and role of mathematical definitions in understanding.

**Response 1:** In some sense, a mathematical definition is an isolated tool obtained from a mass

of concepts which is utilizable again and again. From existing mathematical concepts, when a selection is made by rearrangement so that this rearrangement becomes a useful tool, or, when a part of an existing concept is isolated so that it becomes an entity in its own right, a new object is born. It is characterized completely by its definition.

**Response 2:** Some prefer to start with some examples and then work slowly to an abstract setting. And some much prefer to start with a general abstract definition and find examples as they crop up. I am strongly of the first type and always seek to delineate the abstract concept with an array of examples or else I just can't work with it.

**Response 3:** Mathematicians in the process of setting up a whole universe of mathematical definitions and abstractions to express their ideas in their generality, fail to communicate the original examples, which led to these ideas in the first place. This leads to difficulties in comprehending these new ideas even for some fellow mathematicians. It is no wonder that students struggling to comprehend new mathematical definitions and theorems exhibit great difficulties.

### Discussion and Conclusions

The themes that emerged from the responses of the participating mathematicians led to several conclusions, outlined below. I believe that the emergent themes identified in this study are applicable to most mathematicians and the way they understand mathematical definitions. Nevertheless, the sample size of the mathematicians involved was small and all of them were pure mathematicians. It would be interesting to conduct a similar study with a much larger sample and, perhaps, involve applied mathematicians as well. It is possible that wordings of some questions—specifically questions (1) and (3)—in the questionnaire might have influenced the responses, which in turn might have influenced the conclusions.

#### *Predominant Role of Examples in Comprehending New Definitions*

All participating mathematicians described examples as an important cognitive tool they employ during the process of understanding a new definition. One of the processes employed is commonly known as the *inclusion-exclusion principle*. For example, one of the participating mathematicians stated,

I try to see how the definition will exclude/include examples that I already know. For example, if the

definition is about groups then I would try to see whether it clearly divides the groups that I know into those that fit and those that do not.

In other words, a set of mathematical objects can be viewed as specifying a territory. Other mathematical objects can then be differentiated according to whether they belong to a specific territory or not. Examples and counterexamples can be viewed as an approximation of the form or shape of this territory. Therefore, the more numerous and varied the examples, the finer the approximation.

To illustrate, consider the concept of continuous functions. Typically, the first examples of continuous functions students encounter are polynomial functions, followed by trigonometric, exponential, logarithmic functions, and so on. When one encounters the absolute value function and notices it to be continuous but not differentiable, one gains a better understanding of the territory of continuous functions. Therefore, pedagogically speaking, it helps to have a good supply of examples when learning a new concept or definition. Thus, a definition is a collection of all the examples that conform to the definition. According to one of the mathematicians, definitions cannot stand abstractly without examples. He said, "All the examples which satisfy a definition is like a kernel of the definition."

Definitions in one's area of expertise are easily understood because one has at hand a rich supply of objects on which to test the definition. Thus, examples are like pillars on which definitions are built. For some expert mathematicians, understanding a new definition involves the process of continuously molding the definition so that it approximately fits into their area of expertise.

For example, there is a close relation between algebraic geometry and commutative algebra in that many algebraic objects have parallels in geometry. When a mathematician encounters a definition in algebra (concerning, say, projective module), the mathematician may prefer to view the definition as one in geometry (in this case, vector bundles).

#### *Role of Equivalent Reformulations of Definitions in Enriching Understanding*

At times, a definition and a theorem can be interchanged. That is, the theorem yields an equivalent reformulation of the definition, leading to deeper understanding. For example, an equilateral triangle may be defined as a triangle in which all sides have equal length. The theorem stating that a triangle is

equilateral if and only if it is equiangular provides an equivalent reformulation of this definition. This realization gives new insight into the original definition. Thus, a mathematical object is better understood through characterizing properties.

#### *Role of Evoked Conflicts and Their Resolutions in Improved Understanding*

Sometimes when one's understanding of a definition encounters an example that conflicts with one's understanding, it can lead to transformation in one's thinking that helps to resolve the conflict in understanding. For example, one might believe that all continuous functions are differentiable until one encounters an example that is continuous but not differentiable. It appears that understanding of a definition undergoes constant change as conflicts of various kinds are evoked and resolved. One participating mathematician described how some conflicts simply disappear as one views the definition with a new perspective and at other times by performing a simple calculation.

#### *Role of Definitions in Mathematics*

One of our participants believed the central role of definitions is to *prove* theorems. For example, the importance of understanding the definition of a continuous function is not found in the ability to apply it in order to determine the continuity of individual functions. The importance of understanding the definition is using it to make broader conclusions. For instance, one can use the definition of continuity to prove that all polynomial functions are continuous.

#### *Stages in Understanding Definitions*

The participant responses supported several theoretical models of stages in understanding definitions. In fact, the view of stages represented by Response 4 in "Methods of Understanding" reflects ideas from many of the participants. According to Response 4, understanding a mathematical definition consists of four stages (cf. Manin, 2007). The first stage involves familiarizing oneself with the formalized mathematical language of and the dialect in which it is written. For instance, when one attempts to grapple with the definition of continuous functions, one should be familiar with the notation used to express limit of a function. The second stage involves understanding the definition as a syntactically correct and meaningful expression represented with mathematical language. This second stage also requires development of an intuitive understanding of the

definition, a "semantic cloud." It appears that *semantic cloud* is the mathematical aspect of what is termed as *concept image* (Tall & Vinner, 1981). According to Thurston (1994), learning a mathematical topic consists of building useful, non-formal mental models, and this learning cannot be accomplished by studying definitions and rigorous proofs alone. I include these processes as part of the semantic cloud. Using the representative example of continuous functions, its semantic cloud could consist of ideas of graphs without gaps. In the third stage, one acquires a variety of examples, ranging from "small objects" to "big objects". In the case of continuous functions, small objects could include the constant function, identity function, etc., while big objects could include continuous functions that are nowhere differentiable. In the fourth stage, one acquires the knowledge of how the definition is used in theorems and in other related topics. This stage leads to an understanding of why a particular definition is formulated in a specific way. Also, it often leads to an appreciation of the need to characterize mathematical concepts with precise, unambiguous definitions. This understanding may also lead to the construction of new examples and counter-examples. Hence, the third and fourth stages are dynamically nested, representing the complexity of the learning process for mathematical definitions.

#### **Pedagogical Implications**

Participants provided the following pedagogical recommendations:

**Response 1:** Teaching strategies must take into consideration the different challenges posed by each of the stages (given above) in understanding the mathematical definition.

**Response 2:** It is preferable to start with some examples and then work slowly towards an abstract setting. It is important to delineate abstract concepts with an array of examples which tie the idea into their cognitive framework.

**Response 3:** The teaching strategy should aim to convey that a mathematical definition is just as tangible as a table or a chair. The student should be able to recognize it, use it for the routine purposes for which it is meant, and perhaps use it in a novel way, just as one can recognize a chair, sit on one, or in case of need stand on one to reach a high shelf.

**Response 4:** Solving well-formulated problems is an important strategy to gain in-depth understanding of the definition.

According to one of our participants, there is a wide gulf between *mathematical thinking* and formal *mathematical writing*. While *mathematical thinking* involves creativity unrestrained by demands of structure, logic, and rigor, guided only by intuition, knowledge, and experience, *mathematical writing*, particularly the development of proofs, does not permit such freedom. According to this mathematician, *abstraction* is a very relative term. He concludes:

Abstraction is very individualistic. One question students frequently ask is: What can one say or do about a mathematical object when it is so abstract that it can't be even be seen or imagined? I wish to suggest that a mathematical object is its defining property. The more you learn its characterizing properties the better you get to "know" it.

### Acknowledgments

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## Appendix

### *Questionnaire*

1. How would you comprehend a definition in your area of expertise and a definition in an area less familiar to you? More specifically, is there a specific identifiable intellectual process specific to your individuality which is called up when needed to comprehend a definition. As a part of the process, do you use special examples and then abstract the process, or draw pictures, schematic diagrams, etc.?
2. In the course of your mathematical development, are you aware of any changes in the way you comprehend a new definition? In what ways, if any, are they different from the ones you use now?
3. Do you have a recollection of having understood a definition or a mathematical statement in a particular way which later on resulted in a conflict? If so, how is the awareness of the conflict triggered? Does the awareness occur spontaneously or when working consciously at it?
4. In your experience, if someone recasts your definition in a different way, what method(s) do you use for reconciling or understanding the new definition?
5. Is it possible to evolve general strategies for understanding mathematical definitions based on your research experience or teaching? To what extent are the strategies common or different across the subjects (algebra, topology, geometry, analysis,...).

## Book Review...

# Mathematics Education at Highly Effective Schools That Serve the Poor: A Book Review

Eileen Murray

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Richard S. Kitchen, from the University of New Mexico (UNM), has a long history of work in the mathematics education field focusing on issues of equity, diversity, and multiculturalism in the classroom. Most recently, Kitchen acted as lead researcher on a project initiated in August 2002 with fellow UNM colleagues, Julie DePree, Sylvia Celedon-Pattichis, and Jonathan Brinkerhoff, which explored the characteristics of nine public secondary schools (grades 6-12) that have demonstrated high achievement while serving high-poverty communities. Their book, *Mathematics Education at Highly Effective Schools That Serve the Poor*, presents some of the results of this research.

The nine schools participating in the study had won a larger national competition, the Hewlett-Packard (HP) Company's High-Achieving School Initiative (HAS). These schools received an HP Wireless Mobile classroom (including laptops, a digital camera, and instructional delivery software), a cash award of \$7500, classroom technological support provided by UNM, and professional development opportunities for teachers. Applications from 231 schools from 32 states, the District of Columbia, and Puerto Rico were initially submitted, but only 88 schools were eligible for the competition. Their applications were reviewed by the UNM research team and HP.

Eligible schools demonstrated a free or reduced lunch rate of at least 50% and a sustained exemplary academic achievement for at least three consecutive years. Furthermore, the schools had the technological infrastructure necessary to support an HP Wireless Mobile Classroom. As well as meeting these basic requirements, the 88 finalists provided information about their schools, such as demographic statistics on student race and ethnicity, teacher experience and education, and an analysis of testing data comparing

their schools with others in the same state and the nation. The schools supplied additional characteristics that they felt contributed to their success, such as administrative and parental support, faculty-student ratios, and extra academic support.

Telling the stories of these nine winning schools allowed the researchers to discuss "characteristics as identified by teachers, students, and administrators that distinguished their schools as highly effective in mathematics" and the "teachers' beliefs and knowledge (conceptions) and practices about mathematics curriculum, instruction, and assessment" (Kitchen, DePree, Celedon-Pattichis, & Brinkerhoff, 2007, p. 167). Their goal was to provide a picture of highly effective schools and to determine shared characteristics that might be missing from ineffective schools. Throughout the stories, Kitchen et al. provided detailed examples of specific strategies these schools use that allow traditionally underserved students the chance to receive and use the educational opportunities they need to succeed at high levels in mathematics. This work is important for practitioners, students, and teacher educators in that it helps improve understanding of features that help mathematics educators better serve *all* students.

### Research Findings

The researchers used qualitative methods to identify major patterns and themes that characterized the participating schools. Evidence from school- and classroom-level data included interviews with teachers, administrators, and students, an administrator survey, and classroom artifacts. A classroom observation instrument was used to collect quantitative data that could determine the extent to which students experienced reform-oriented instruction. Using an iterative coding process with the qualitative data, the research findings were first grouped into three major themes: "(a) high expectations and sustained support for academic achievement, (b) challenging mathematical content and high-level mathematics instruction that focused on problem solving and sense

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making (as opposed to rote instruction), and (c) the importance of building relationships” (Kitchen et al., 2007, p. xiv). During analysis, the researchers identified areas related to each major theme. In order for a theme or related area to be included in the analysis, it had to be found in at least two teacher interviews at more than half of the participating schools.

For the second theme of challenging mathematical content and high-level instruction, the researchers explored five related areas: the prioritization of problem solving, the completion of an Algebra I equivalent by 8th grade, students’ mathematical communication and engagement in inquiry, mathematics curriculum as a work in progress, and preparation for success on standardized tests while teaching beyond the test. Of the major themes, I will focus on how the book describes this second theme of challenging mathematical content and high-level instruction, and, in particular, its related area of the prioritization of problem solving.

Most of the participating schools had the goal of preparing their students to be successful on state and national tests, and the teachers purposefully structured their mathematics curriculum to prepare their students to be successful on these standardized tests. However, standardized tests did not determine the mathematics curriculum and instruction, and this is evident in teachers’ decisions to develop students’ critical thinking and problem solving, going beyond skills-based curriculum and instruction. Evidence from the interviews convincingly shows how the teachers go beyond test preparation.

In addition to these similarities, Kitchen and colleagues discussed the differences between schools in terms of the teachers’ use of skills-based instruction and their focus within lessons on mathematical problem solving. Using data from lesson observations and interviews, the researchers concluded, “the focus at the highly effective schools was teaching a challenging mathematics curriculum that developed students’ critical thinking capacities through problem solving” (Kitchen et al., 2007, p. 163). However, even though all the schools are considered to have such mathematics programs, some schools are more aligned with a standards-based curriculum and pedagogy, while others rely more on skills-based instruction. This variation makes it clear that there is no single way to be successful.

In working to understand the relationship between skills-based instruction and problem solving, the researchers deduced that the teachers at the schools had

to be flexible in their philosophical orientations about the nature of mathematics. This conclusion is particularly interesting because it could be seen to contradict other literature concerning teachers’ beliefs about the nature of mathematics. Specifically, in the reviewed book there are quotes from the teachers indicating that some of the teachers have a linear view of mathematics. In linear (sequential or hierarchical) learning, students move from lower- to higher-order cognitive tasks. Since higher-order cognitive tasks are more complex, they require deeper content understanding. Understanding of mathematical concepts evolves from simple, disjoint ideas to complex, connected ideas: Before students are able to tackle more advanced topics they must have mastered prerequisite, basic skills. When teachers hold this view, student-teacher interactions usually focus on empirical and procedural issues instead of critical thinking (Raudenbush, Rowan, & Cheong, 1993). This view can create issues with teaching critical thinking and problem solving in the classroom. Specifically, teachers with more traditional views, i.e. linear, are more likely to think that higher order thinking tasks are not appropriate for all students. For example, Zohar, Degani, and Vaaknin (2001) found that one of the major factors in teachers’ decision to not use critical thinking-based learning with low-achieving students is the belief that higher-order thinking is inappropriate for these students. This belief is directly related to the teachers’ views of teaching and learning. Therefore, there seems to be a contradiction between what the teachers who seemed to hold a linear view of mathematics believed and how they taught, since they still implemented problem solving in their classrooms.

Fortunately, the authors clarified this issue by discussing how the three dominant characterizations of mathematics knowledge (instrumentalist, Platonist, and problem solving) are not mutually exclusive. That is to say, the teachers utilized skills-based instruction as well as problem solving in their classrooms. They believed that both were necessary for success in secondary mathematics. Though all teachers professed to value problem solving in interviews, some teachers displayed a teaching style more closely aligned with standards-based instruction, by using higher levels of mathematical analysis and discourse. Others continued to emphasize drill-focused pedagogical strategies. Kitchen et al. explained these inconsistencies by reminding the reader that the teachers’ primary goal was to do whatever necessary to serve the needs of the students. The teachers must have flexible philosophies

allowing them to use many different instructional techniques.

In attempting to account for variation in teachers' focus on problem solving, Kitchen and his colleagues presented differences between the schools that could impact teachers' pedagogical decisions. In each of the nine schools, teaching and learning was a primary value in the school culture. Administrators and teachers thought seriously about discipline and "a majority of the schools had discipline policies reinforcing the notion that learning was the top priority and obstructing the learning of others was a serious offense" (p. 148). In three of the participating schools, students signed a contract in which they agreed to strict behavioral norms. These schools had longer school days, mandatory Saturday classes, and summer school. Teachers committed to extended workdays, extra tutoring time, Saturday classes, and summer school, as well as being available by phone to students after hours. The discipline policy and support services promoted the goal of positively impacting student learning and achievement. These components of the schools allowed the teachers to commit to supporting the students in learning challenging mathematics, including critical thinking and problem solving. All of the teachers in the study felt that problem solving was an important component of successful mathematics instruction, but outside influences caused them to implement particular pedagogical strategies to varying degrees. Hence, though there was some contradicting evidence between teacher intentions and their actions in the classroom, the researchers did a good job of explaining these differences and showing why they could have occurred. This added to the strength of the study and provided additional support for some of Kitchen's previous findings (Kitchen, 2003).

### **Comparison to Other Research**

#### *Teacher Support Structures*

In one of Kitchen's earlier studies, he looked at teachers' abilities to implement standards-based curriculum in their classrooms. Through work with secondary teachers in a summer institute, Kitchen found that "teachers' overwhelming workload served as the primary barrier to reforming their classroom practices and implementing innovative instructional strategies" (Kitchen, 2003, p. 3). The teachers did not have the time or energy they needed to develop new ways of teaching. They did not have support from administrators, colleagues, or parents. Therefore, the

fact that the highly effective schools in the present study have both administrative and parental support, as well as cooperation among teachers, lends more credence to the conclusion that these types of support increase the likelihood of a school being effective.

#### *Indicators of Effectiveness*

Kitchen and his colleagues' definition of effective schools is restrictive. The participating schools needed to show "sustained exemplary academic achievement, particularly in mathematics, over a minimum of from 3 to 5 consecutive years across a variety of indicators" (Kitchen et al., 2007, p. 22). The indicators included high scores on standardized tests as compared to other schools in their district, state, and nation; standards-based curriculum and instruction; use of alternative assessments; and a high percentage of students matriculating in advanced placement courses. Other research focusing on schools serving the poor use different indicators to evaluate school success.

As an example, Boaler and Staples (2008) did a longitudinal study of three schools that all shared a common characteristic of retaining a committed and knowledgeable mathematics department. Teachers at one of the schools, Railside, used a reform-oriented curriculum focusing on conceptual problems and mixed ability group work. Student achievement data showed that the Railside students started at statistically significant lower levels than the other schools. However, after two years, the student achievement data showed a statistically significant difference in favor of the Railside students. Nevertheless, the Railside students did not fair as well on the state standardized tests as the other schools. The authors believed that this had to do with cultural and linguistic barriers on such assessments.

The reason this work is of interest is that Railside would not necessarily be considered for Kitchen's study because the students did not perform well on standardized tests. However, the students at Railside had a more positive attitude towards mathematics, did better on curriculum-aligned tests, and had a smaller achievement gap between students of different ethnic and cultural groups. Furthermore, the Railside teachers showed many characteristics prevalent in Kitchen's study such as holding high expectations for students and providing challenging, standards-based curriculum. Therefore, it is important that the readers acknowledge the diversity of schools in the United States that do successfully serve diverse populations. Kitchen's sample represents only a particular set of



“highly effective schools serving high-poverty communities across the country” (Kitchen et al., 2007, p. 165) satisfying a specific set of criteria set by this particular research team. Fortunately, the authors recognized this limitation and urged readers to consider investigating other schools located in less affluent communities to see if the findings from this study hold in other contexts.

### Conclusions

Kitchen and his colleagues’ study set out to identify significant characteristics that contributed to the success of secondary schools serving high-poverty communities. The researchers also explored teachers’ beliefs and practices related to mathematics teaching and learning. The results of their analysis provide the reader with numerous examples of effective strategies, such as high expectations for students, support for academic achievement, challenging mathematical content, and high-level instruction. They found that, as long as teachers and administrators were committed to doing all that was necessary to help their students learn deep mathematical ideas and achieve at high levels, their students would deliver.

This book is a powerful example of how schools can attend to traditionally marginalized populations effectively when they are committed to serving and understanding the needs of their students. Kitchen and his colleagues did a wonderful job of providing evidence of the characteristics present in the schools,

as well as pointing out and discussing possible discrepancies and limitations of their findings. Anyone in the mathematics education community, including teachers, administrators, students, and teacher educators, will find the description of these schools useful in furthering their thinking about how to become an effective teacher or school. With the U.S. student population becoming more diverse, but the teacher pool becoming more homogeneous, Kitchen et al.’s work is especially timely for the mathematics education community.

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<p>AMESA Seventeenth Annual National Congress <a href="http://www.amesa.org.za/AMESA/">http://www.amesa.org.za/AMESA/</a></p>	<p>Johannesburg, South Africa</p>	<p>June 27–July 1, 2011</p>
<p>ICME12 International Congress on Mathematical Education <a href="http://www.icme12.org">http://www.icme12.org</a></p>	<p>Seoul, Korea</p>	<p>July 8–15, 2012</p>
<p>PME-34 International Group for the Psychology of Mathematics Education <a href="http://pme34.lcc.ufmg.br/">http://pme34.lcc.ufmg.br/</a></p>	<p>Belo Horizonte, Brazil</p>	<p>July 18–23, 2010</p>
<p>JSM of the ASA Joint Statistical Meetings of the American Statistical Association <a href="http://www.amstat.org/meetings/jsm/2010/">http://www.amstat.org/meetings/jsm/2010/</a></p>	<p>Vancouver, BC, Canada</p>	<p>July 31–August 5, 2010</p>
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<p>SSMA School Science and Mathematics Association <a href="http://www.ssma.org">http://www.ssma.org</a></p>	<p>Ft. Meyers, FL</p>	<p>November 4–6, 2010</p>
<p>AERA American Educational Research Association Annual Meeting <a href="http://www.aera.net">http://www.aera.net</a></p>	<p>New Orleans, LA</p>	<p>April 8–12, 2011</p>
<p>PME-NA North American Chapter: International Group for the Psychology of Mathematics Education <a href="http://pmena.org/2010">http://pmena.org/2010</a></p>	<p>Columbus, OH</p>	<p>October 28–31, 2010</p>
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<p>AMTE Association of Mathematics Teacher Educators <a href="http://amte.net/conferences">http://amte.net/conferences</a></p>	<p>Irvine, CA</p>	<p>January 28–29, 2011</p>
<p>NCSM National Council of Supervisors of Mathematics Annual Conference <a href="http://www.mathedleadership.org/events">http://www.mathedleadership.org/events</a></p>	<p>Indianapolis, IN</p>	<p>April 11–13, 2010</p>
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