Teaching Mathematics Through Spreadsheet-Oriented Problem Solving

Sergei Abramovich and Gary Brown

An effective philosophy in the teaching of mathematics is to provide a context in the classroom in which students can learn insightfully rather than mechanically (Stanic & Kilpatrick, 1989). For example, a class may be presented with a realistic and relevant problem. The problem serves as a stepping stone for each student to begin a journey of exploration and inquiry. The role of the teacher is to provide a map for the students and to guide them toward certain key landmarks. Because a map can never take the place of the actual journey (Stanic & Kilpatrick, 1989), the strength of this approach is that it encourages each student to start on a journey of problem solving. Along the way, students discover concepts and develop skills in the context of their own experience through their own insights.

One powerful environment for this journey of exploration and extension of a problem is a spreadsheet. This paper provides an example of how an insightful approach to learning elementary concepts of number theory combined with the use of a spreadsheet can serve as an effective method by which concepts are discovered and skills are developed. With this approach, students can experience the real power of the theory of numbers rather than just learn definitions and rules (Reys, Suydam, & Lindquist, 1992). Figure 1 provides an overview of the method, showing the key landmarks the teacher will want to be sure the students reach along this journey of learning.

The journey begins when students are presented with the problem shown in Figure 2. Students can then be put into groups of three or four and asked to explore various approaches to solving this problem. During this time, the teacher can ask questions and make comments to assist students in pursuing a problem solving strategy. For example, if a group is unsure what the question is asking, the teacher might suggest that the students reformulate the question in their own words; if this proves helpful, they can present their paraphrase to the rest of the class. Once the problem is understood, the first step the teacher will want the students to reach is the idea of making a chart.

![Figure 1: Overview of the general solution process.](image-url)

If a group does not think of making a chart the teacher can suggest that one way of approaching a difficult problem is to create a similar type of problem with simpler numbers (or no numbers) to which the answer is already known and then apply the same general method of solution to the more difficult problem. The teacher may then ask, “Can any of the groups invent an analogous problem - one that is similar but that you can solve more easily?” Often, this is not an elementary task. For instance, the students may discover that if the analogous problem is not similar enough in structure to the original, the method of obtaining its solution might not parallel the method required for the original. Also, the students may find that an analogous problem that is too trivial is solved with “common sense” and causes the actual method of solution to be unclear, while one that is not trivial enough leads to the same difficulties as the original problem. Creating analogous problems helps students develop problem solving skills. A good example of such an analogous problem is provided in Figure 3.

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Sergei Abramovich is a visiting professor in the Department of Mathematics Education at the University of Georgia, where he teaches a variety of courses using computers. His research interests include the study of effective use of technology-rich environments in mathematics teaching and learning.

Gary Brown is currently enrolled in the M.Ed. program at the University of Georgia and holds a B.S. in psychology from Kennesaw State College. He plans to teach secondary mathematics upon graduation, and his research interests include cognitive issues and the use of technology in mathematics education.
Given this problem, students might think of making a chart because of their familiarity with calendars. The teacher may then suggest this strategy to any group that did not think of making a chart for the original problem. Once all groups have constructed some type of chart, sharing their ideas on the board can provide a fruitful time for discussion about the strengths and weaknesses of each chart. Students often gain valuable insight by communicating not only their ideas but also the cognitive processes which led them to those ideas. Convincing others of the validity of one's ideas is an important part of the discourse.

Once the groups have each developed some type of chart, a context exists in which students can see the advantages of a spreadsheet. The teacher can first change or extend the original problem by asking questions such as:

- What if you also wanted to know which days you would be tutoring 3 students?
- What if the days were 3, 4, and 5 instead of 2, 3, and 4?
- Most of these charts only show up to 10 days. What about days 11-20? 21-30?

The students will then have to erase, edit, and rework their charts. The teacher may continue to change and extend the problem until the students realize the frustration caused by having to repeatedly edit and recalculate their previous work. The spreadsheet can then be introduced as an alternate way to chart the data. As the teacher begins to demonstrate the use of the spreadsheet, initially students might suggest that all data be entered directly into each cell, much like a chart one would produce with pencil and paper. Although such a chart created with the spreadsheet is more visually appealing, functionally it provides no significant advantage over a handwritten chart (see Figure 4).

So once again the problem can be changed and extended to demonstrate that the editing and reworking of the chart on the spreadsheet is equally cumbersome. Then the teacher may say, "It requires a lot of work to change every single value if we make even the smallest adjustment to the original problem. Maybe there is a better way to make a chart...” Finally, the context has been created into which the teacher can introduce the concepts of generalization, formulas, and variables through the spreadsheet.

Initially, students will probably not think of generalizing or writing formulas; however, it is important that the teacher does not simply provide the formulas directly. The teacher should merely guide the students in order that they might discover the concept of generalizing on their own. An example of such guidance is provided in the following dialogue between teacher and student:

T: What patterns do you see in column A?
S: The numbers get bigger.
T: Exactly. Now these numbers also get bigger (the teacher writes the numbers 1, 4, 18, 44, 100, 580). Is there a difference?
S: Yes. The numbers in column A get bigger by exactly every time.
T: That's right! So tell me what number goes in row 10?
S: That's easy. It's 9.
T: Great! But how did you know that it was 9?
S: Because 9 comes after 8.
T: But 100 also comes after 8.
S: Oh, I see - like before. It's because 9 is exactly 1 bigger than 8.
T: Very good. Can you state that as an equation?
S: 9 = 8 + 1.
T: There you go! Now, let's say that cell A100 contains a number N. What would be in A101?
S: It would be N plus 1.
T: Perfect. So if you enter a 1 into A2, what formula can you put into A3?
S: A2 plus 1.
The teacher could then enter the formula into cell A3 and copy it down. At this point, it would be helpful to toggle between displaying the values in column A and displaying the formulas in column A.
(see Figure 5). Thus the spreadsheet provides a powerful environment in which visualization of abstract concepts such as variables or formulas can lead to a higher level of understanding.

After developing a formula to generate the numbers in column A, students can be asked to develop a formula for column B. The general formula for columns B, C, and D is the following:

$$=IF(INT($A2/B$1)=A2/B$1,"*","\").$$

The path to this general formula is rich with opportunities for enlightenment and enrichment of some basic mathematical concepts. The discussion below demonstrates a select few of the discoveries that can be made along the way.

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T: Can you state in words a formula for column B?
S: If the number in column A is even, then put a mark in column B. If the number in column A is odd, then leave column B blank.
T: That's good. But the spreadsheet doesn't know what even' or 'odd' means. How else could you state it?
S: Well, a number is even if it's divisible by two.
T: What do you mean by divisible?
S: 2 goes into it.
T: But 2 goes into seven... 3.5 times.
S: I mean that when you divide it, it divides evenly.
T: What do you mean by evenly?
S: There is no remainder.

Figure 5: Visualization of the ability within the spreadsheet to toggle between displaying the calculated values within cells and the formulas which generated those values.
T: Exactly. So now tell me what an even number is.
S: A number is even if you divide it by 2 and get no remainder.
T: Excellent. Now, let me ask you this...

Though this discussion is only one of many possible paths to the general formula, the concepts that naturally arose included a deeper understanding of the definition of even/odd numbers and divisibility. Other concepts that could be discussed include multiplicity, the greatest common divisor, and the least common multiple. Although most students will already be familiar with these concepts, they will see - perhaps for the first time - some connections between them and how they are used together in a real-world context rather than merely as a series of discrete facts to be memorized and used mechanically.

Other initial approaches to generate column B may not lead directly to the general formula, but because it is the process rather than the final product which leads to understanding, even a pursuit that results in a wrong answer can lead to understanding. For example, a student could have suggested that one way to mark column B is the following: “If the previous cell is blank, then mark that cell; if the previous cell is marked, then leave it blank.” Thus, a formula such as =IF(B2=“""""","""") would go into B3 and could be copied down (cell B2 would be left blank). Although the teacher would know that this is not necessarily a ‘wrong’ approach, he/she might be tempted to quickly dismiss the possibility of pursuing it because he/she would know that it is not the better formula (because it is not easily transferable to columns C and D). However, rather than tell this to the students, the teacher can allow them to discover this themselves by having the groups present their formulas for column B. Once they are asked to create formulas for columns C and D, students will recognize that some formulas (such as the general formula shown previously) can be easily generalized for any given number of days, while others are more difficult to generalize for any number of days. Thus, students should see that although there may be several acceptable formulas, one formula may be better or more efficient for a given situation. Here, the teacher could state that formulas which may not be the best for this specific application may, in fact, be much better than the other for some other applications. A challenging question might be, “Can you think of a type of chart where the less general formula might be favored over the more general formula?”

At this point, the general equations for columns A through D should be entered and the students should now be asked to create a formula for marking the days in which they would tutor no students. After this is accomplished, the spreadsheet will look like the one shown in Figure 6.

At this point, the original problem is solved. If this were a handwritten chart, it would be time consuming to change or extend the problem. However, the spreadsheet provides a powerful environment in which new concepts can be explored and extended without spending valuable time editing and recalculating. As was shown in the overview diagram in Figure 1, one method of accomplishing this systematically is to first explore other questions within the solved problem, then to extend the original problem, and explore again. Thus a pattern of extension followed by exploration may be continued indefinitely.

First, to explore the information gathered through the original problem, the teacher might ask, “What do you notice about the data in column E?” (given the spreadsheet shown in Figure 6). Some students might notice that the numbers are in pairs. The teacher could then ask whether
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Figure 7: The completed spreadsheet for the original problem and three possible extensions of the problem.
this continues for all numbers and, because of the versatility of the spreadsheet, could actually demonstrate quickly that it does, in fact, continue. Students could also be asked whether there are any numbers that stand alone. Students should respond that the number 1 is by itself. A challenging question might then be, “Why do you think the number 1 is the only number that is not part of a pair?”

Another relationship that students may notice in column E is that the numbers are all odd. From this observation students may see that the numbers appear to be all prime numbers. Here, the concepts of primes and prime twins as they occur in real-world situations can be discussed (Abramovich, 1994). The students can then observe that this pattern continues only up to the 25th day. Thus the natural question arises: “Which numbers in column E are not prime?” Once again, the spreadsheet provides quick access to what would normally be a lengthy task. The numbers in column E that are not prime are 25, 35, 49, 55, etc... A worthwhile investigation might be to discuss what these numbers have in common. It should be clear that this process of exploring and questioning is virtually endless. A wonderful result from creating such an open-ended environment is that each group can pursue its own ideas, which allows for greater motivation as well as ownership of the concepts discovered. In fact, within this type of a learning environment it is quite common for students to discover things that the teacher has not discovered yet. This creates an image of mathematics as a lab science, and may attract students to careers in mathematics (Steen, 1991).

After exploring the original data, the teacher can then extend the problem to broaden the environment in which to explore. For example, the teacher might pose questions such as:
- On which days will you tutor all 3 students?
- We can see in column E that you will never have 2 days off in a row. Is there any three day period when you will be off for 2 of the 3 days?
- Can we generate a formula for marking only prime numbers?
- How many students will you tutor for each given day?

An example of the resulting spreadsheet from these three extensions is shown in Figure 7.

Once again, students can explore the new data for various relationships. The teacher may guide their discovery by asking questions such as “Do the numbers in column F have anything in common?” or “Do you see any patterns in how the numbers in column F are related to those in any of the other columns?” These questions can lead to discussions of many concepts which may again include multiplicity, divisibility, the greatest common factor, and the least common multiple. There are many visual patterns in this one example alone. One interesting pattern can be suggested by asking students, “How often is the number in column F divisible by 10?”

Students will quickly see that this occurs in cycles of 5. This is rather interesting because it occurs also in column G.

Another interesting question might be (for a given column of data), “What could be a real life example of why you might want to know this?” This allows students to make connections between the concepts discovered and the real world. There is also much to be discovered regarding the symmetry of numbers in various columns. For example, there exist several lines of symmetry in

Figure 8: Summary of several concepts and skills that can be discovered and developed in solving the given problem within the environment of a spreadsheet.
column I about which the numbers are identical. It is also interesting to note that this symmetry occurs about the values in columns E, F, and G. Clearly, there appears to be no limit to the depth of exploration and the breadth of extension possible using the spreadsheet.

The example provided demonstrates the ample possibilities in the classroom for insightful learning through exploration and extension of a problem within the environment of a spreadsheet. This environment for learning readily facilitates open-ended questions and provides a favorable platform for the students to discover concepts and to develop skills. Computer-enhanced exploration also serves as a more natural and applied context for discourse and learning than the contrived environment of a textbook or lecture. Moreover, solving realistic problems opens the door to see “what children can do when an attempt is made to link subject matter to experience” (Stanic & Kilpatrick, 1989, p. 20). Figure 8 provides a summary of the spreadsheet-paved journey filled with examples of particular concepts that can be discovered and general skills that can be developed along the path of exploration and extension.

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