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A Note from the Editor
Dear TME Readers,

I would like to present the final issue of Volume 15 of *The Mathematics Educator (TME)*. It was fifteen years ago when a group of nine seminar participants created the first TME publication. With the support of the Mathematics Education Student Association and the Department of Mathematics Education at the University of Georgia (UGA) these students laid the foundation for what is now a nationally and internationally known mathematics education journal.

The growth that *TME* has experienced over the past fifteen years is noteworthy. First and foremost, the *TME* readership and submission rates have increased each year. To keep up with this growth and to make the journal more accessible to a wider range of readers, TME moved to an online format in 1998. Recently, *TME*’s website was revamped in order to provide easier access to all of our previous issues. As a final note with regard to the growth of *TME*, I want to acknowledge the publication of our first monograph issue this year. Similar to the first publication of *TME*, the monograph was the result of a seminar developed by UGA graduate students; and the monograph on equity was the result of their endeavor to share their thoughts with the larger mathematics education community about critical issues within mathematics education.

This issue keeps with the *TME* tradition of publishing articles with diverse views from university faculty members, graduate students, and practicing teachers about a range of topics. Andrew Izsák—a faculty member at UGA and a winner of an Early Career Publication Award from the American Educational Research Association’s Special Interest Group on Research in Mathematics Education—kicks off the fall issue with a guest editorial in which he highlights his perspective about framing research studies. The second piece, by Mine Isiksal, poses a question about the effects that gender and year in a teacher preparation program have on pre-service mathematics teachers’ performance and mathematical self-efficacy. In the third piece, Patricio Herbst, Gloriana Gonzalez, and Michele Macke bring to light some of the reasons that students have difficulties with definitions and present a game that has the potential for improving student understanding. In the fourth piece, Jon R. Star and Amanda Jansen Hofmann report on a comparative study investigating students’ epistemological conceptions of mathematics. The final piece, written by Rongjin Huang and Frederick K. S. Leung, presents a case study of a Chinese mathematics classroom in which they explore the ideas of teacher-centered and student-centered classrooms. In their invited article, they push the mathematics education community to reconsider what constitutes a student-centered classroom.

I hope that my brief descriptions of the articles published in this issue spark your interest to read further. I also encourage readers to consider contributing to *TME* by reviewing manuscripts, submitting manuscripts, or by joining our editorial team. Finally, I wish to thank the many contributors of this particular issue and the authors and editors who have worked conscientiously and diligently to prepare articles that provoke interest and thought.

Sincerely,

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About the cover
Cover artwork by Thomas E. Ricks. The Triumph of 15 Years and Counting...
For questions or comments, contact: tomricks@uga.edu

Triumph of 15 Years celebrates the 15th consecutive year of publication for this journal. It is an artistic acknowledgment of the tremendous cooperation, dedication, and commitment of the many people involved in its success. *TME* is entirely the result of volunteer effort. Producing and maintaining a rigorously peer-reviewed journal is no small task, and the artist wishes to congratulate the contributors, reviewers, editors, and other personnel involved in its successful germination and continued growth.

This publication is supported by the College of Education at The University of Georgia
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Learning how to frame research is arguably the most important capacity for new researchers to develop. This is true both for doctoral students and for junior faculty as they establish research programs at new institutions and begin to help students conduct their own research. I take framing research to mean the construction of connections among research questions, the literature(s) within which those questions are positioned, theoretical perspectives or frameworks, and methods that generate warrants sufficient for making convincing claims. Making strong connections among all of these components is essential for building coherent arguments at the center of high quality research in mathematics education (Lesh, Lovitts, & Kelly, 2000; Simon, 2004). Examples of important contributions to our field that have relied on innovative combinations of questions, theoretical perspective, and methods include Schoenfeld’s (1985) work on problem solving and Steffe’s (e.g., Steffe, 2001; Steffe, von Glasersfeld, Richards, & Cobb, 1983) work on children’s construction of whole and rational numbers.

Some recent publications have suggested that doctoral programs in education may not always be producing graduates with the necessary capacities for framing research. Boote and Beile (2005) sampled 30 dissertations from three state-funded colleges of education in the United States, found a wide range in the quality of the literature reviews, and reported that some of the literature reviews were little more than “disjointed summaries of a haphazard collection of literature” (Boote & Beile, 2005, p. 9). Moreover, weak literature reviews may be just part of a larger challenge for new researchers: Schoenfeld (1999) asserted that beginning researchers often do not learn what it means to make and justify claims about educational phenomena or how to frame workable research problems. Part of the challenge is that framing research is a creative process for which there are no formulae.

I struggled in graduate school to understand how authors came to pose the questions with which they began published articles and wondered if I would ever be able to articulate such questions. Perhaps others have had similar experiences. The purposes of the present article are to help doctoral students construct initial images of processes involved in framing research and to serve as a catalyst for reflection among junior faculty who, like me, are learning to support students conducting their own research. I will do this by describing how I framed my own dissertation study at a level of detail often omitted or left tacit in published articles. My dissertation was based on detailed analyses of videotaped interviews, but I hope my general points will help others to embark on other kinds of research projects as well.

An Example of Framing Research

For my dissertation, conducted under Alan Schoenfeld at the University of California, Berkeley, I constructed a theoretical frame for explaining how pairs of eighth-grade students constructed knowledge structures for modeling with algebra a physical device called a winch. The winch (see Figure 1) exemplifies situations that can be modeled by pairs of simultaneous linear functions, a core topic in introductory algebra courses.

![Figure 1. The winch. From “Inscribing the Winch: Mechanisms by Which Students Develop Knowledge Structures for Representing the Physical World with Algebra,” by A. Izsák, 2000, The Journal of the Learning Sciences, 9, p. 33. Copyright 2000 by Lawrence Erlbaum Associates, Inc. Reprinted with permission.](image-url)
The device stands 4 feet tall and at the top has a rod with a handle for turning two spools, one 3 and one 5 inches in circumference. Fishing line attaches one weight to each spool. I will refer to these as the 3-inch weight and 5-inch weight, respectively. Turning the handle moves the weights up and down a yardstick, allowing measurements of heights, displacements, and distances between the two weights. The theoretical frame highlights coordination and reorganization of knowledge for generating, using, and evaluating algebraic representations (Izsák, 2003, 2004).

Getting Ideas for Feasible, Significant Contributions

I entered my doctoral program interested in how students made sense of external representations of functions and began reading relevant literature. I was fortunate because by the early 1990s there were a handful of excellent recently published literature reviews (Kieran, 1992; Leinhardt, Zaslavsky, & Stein, 1990) and edited books (Janvier, 1987; Romberg, Fennema, & Carpenter, 1993) that addressed students’ understandings of functions and the roles of external representations in mathematical thinking. These sources provided an overview of existing findings and extensive bibliographies that suggested the main researchers whose work related to my emerging interests. At the same time, my interests were also being shaped by numerous conversations with faculty and students, courses, and participation in research groups that, among other things, allowed me to observe more advanced doctoral students frame their dissertation projects.

Much of the work on students’ understandings of representations of functions documented difficulties (often termed misconceptions) that students exhibited when solving equations, solving word problems, and interpreting graphical representations. Taken together, these studies made clear that students struggled with standard or normative representations. The literature also contained several theoretical accounts of how students might learn in this domain. One family of closely related, highly visible accounts described process understandings of functions being reified or encapsulated into object understandings (e.g., Kieran, 1992; Sfard, 1991, 1992; Dubinsky & Harel, 1992). Researchers often relied on cross sectional data as evidence for process-object accounts of learning and none presented empirical evidence of reification or encapsulation processes occurring in actual students. Clearly, studies that captured how students might learn to represent functions and solve problems would contribute to the field, but was capturing such phenomena feasible? Perhaps this problem was too difficult for a dissertation project.

A number of faculty and doctoral students at the University of California, Berkeley shared my interest in the role of external representations in mathematical thinking and pointed me to a small number of then recent studies (diSessa, Hammer, Sherin, & Kolpakowski, 1991; Hall, 1990; Hall, Kibler, Wenger, & Truxaw, 1989; Meira, 1995, 1998) that demonstrated students’ capacities to construct their own perhaps non-standard graphic, algebraic, and tabular representations of functions in the course of solving problems. These studies suggested both that past research had overlooked students’ latent capacities for constructing representations and that capturing examples of students learning to represent functions and solve problems might be feasible. In particular, when analyzing interview data in which one pair of students worked with a device similar to the winch, Meira (1995) found a complex interplay between the students’ understanding of the device and of a table that they were developing to represent that device. These data and the analysis were unlike any I had seen in the literature summarized in the previous paragraph. Subsequently, I discovered that other researchers (Greeno, 1993, 1995; Moore, 1993; Piaget, Grize, Szeminska, & Bang, 1968/1977) had also gained detailed access to young children’s, middle school students’, and high school students’ implicit and explicit understandings of linear functions. Something about the winch was engaging to students and perhaps, with the right combination of tasks and students, I could capture data in which students learned to represent functions and solve problems. Furthermore, examination of previous studies suggested three categories of questions that I could pose to students with various initial winch set-ups:

1. Predict the distance between the weights after an arbitrary number of cranks.
2. Determine whether and, if so, when one weight would ever be twice as high as the other.
3. Determine whether and, if so, when the weights would meet at the same height.

Finally, I needed a provisional theoretical lens or lenses to help me identify instances of learning. The existing literature suggested at least two options. One was to look for instances of reification or encapsulation as described in the process-object perspective mentioned above. Another was to look for the genesis of knowledge structures similar to those described in
some recent research on representations of functions (e.g., Schoenfeld, Smith, & Arcavi, 1993). The latter studies documented instances where complex coordination of multiple pieces of prior knowledge was a dominant feature of learning. I did not adopt any one perspective a priori but rather began with a theoretical tool kit that might, or might not, suffice to explain any captured instances of learning. At this point, I was asking how might students learn to represent functions and solve problems, and I had some theoretical tools and a strategy for gathering data that were good enough to get me started.

**Successive Approximations**

I interviewed 14 pairs of students, 6 in spring 1996 and 8 in spring 1997. In most cases, I conducted four or five hour-long interviews with each pair. Because I was interested in capturing instances of learning, I needed students for whom the winch tasks were neither too easy nor too hard. Thus, I began in spring 1996 with a range of students. I interviewed 2 pairs of eighth-grade students from one school taking a pre-algebra course, 2 more pairs of eighth-grade students from the same school taking an Algebra I course, and 2 pairs of tenth-grade students from a different district who had completed an algebra course. At the beginning of the first interview with each pair, I read instructions that simply pointed out the handle, weights, and yard stick on the winch, a set of questions about the device, and scratch paper should the students need any. I made no mention of linear functions or of any representations. As the students worked, I intervened on occasion to clarify my instructions, to ask for further explanation of some comment, or to discuss possible strategies for making progress when students seemed stuck. Students could repeat actions with the winch as often as they liked and worked on problems until they reported satisfaction with their answer. Occasionally I moved students on to the next question when they appeared bogged down to the point of frustration.

Several results from the first round of data collection allowed me to refine how I framed my research. First, the tasks seemed particularly engaging for the Algebra I students. These students often made connections between their algebra course and the interview activities after recognizing that turning the handle generated patterns in heights of weights and distances between weights. Especially promising, given my research question, were examples in which these students constructed apparently novel, yet sensible, equations after struggling to generate expressions and equations that represented height and distance patterns. The pre-algebra students and tenth-grade students evidenced less potential for learning when working on the winch tasks: The pre-algebra students tended to approach all the problems by making tables, and the tenth-grade students could set up and solve equations much more readily. Second, one pair of Algebra I students apparently reasoned about equations in ways consistent with both the process and the object perspective as described in the literature, yet struggled to coordinate their understandings of their representations and the winch. This suggested that the process-object account was insufficient for explaining how students might learn to represent situations that could be modeled by linear functions. Third, detailed analysis of this same pair suggested that they had constructed a new knowledge structure resembling one described by Sherin (2001).

Analyzing these data led me to frame my research with greater precision: I focused my attention on how Algebra I students could construct knowledge structures for modeling the winch with algebra and took initial steps at creating theory for explaining such phenomena. The theory proposed two learning mechanisms, notation variation and mapping variation, that described processes by which students refined and coordinated their algebraic expressions and the correspondences they established between parts of those expressions and features of the winch (see Izsák, 2000, for details). At this point, I needed further examples of students’ constructions and my committee was intimating that I should attempt “deeper” theory that explained in greater detail how students drew on their existing knowledge.

In spring 1997 I interviewed 7 more pairs of Algebra I students and 1 more pair of pre-algebra students using essentially the same set of tasks. The students came from the middle school used in the first round of data collection. As I had hoped, the second round of data collection produced further examples in which students constructed apparently novel, yet sensible, equations after struggling to represent height and distance patterns on the winch with expressions and equations (Izsák, 2003, 2004). One of the most interesting episodes occurred when one pair of students examined the winch set up so that the 3-inch weight started by the 14-inch mark and the 5-inch weight started by the 0-inch mark. Both weights went up when the students turned the handle, and the question was: Can you predict how far apart the weights will be as you turn the crank? If so, how? The students treated all distances as positive and struggled to represent a
pattern that first decreased to zero and then increased. Eventually, one of the students generated a pair of expressions \(14 - 2n\) and \(2n\) where \(n\) represented number of cranks. She used the first expression to calculate distances before the weights met and the second to calculate distances after. In so doing, she counted \(n\) from two different starting points. Her partner understood that computations made with these expressions matched distances measured on the winch, yet rejected the pair of expressions as a legitimate representation.

As the students continued to work, they evidenced a range of criteria for evaluating algebraic representations. Two of the several examples were *single equation*, the criterion that single equations are better than multiple ones, and *consistent interpretation*, the criterion that the number of cranks always be counted from the same starting point. The students continued to work until they generated \((n - 7) \times 2l = d\), an equation that correctly predicted distances between the weights and simultaneously satisfied all the criteria they had mentioned (Izsák, 2004). These data made clear that the students’ knowledge for evaluating algebraic representations fundamentally shaped the direction of their work. Furthermore, analysis of subsequent interviews with the same pair of students suggested that they had constructed a new knowledge structure (Izsák, 2004). Thus, criteria apparently played a role in learning.

At this point, I was confident that my data were sufficient for making legitimate progress on my research question, but how I framed my study continued to evolve through further analyses of both data and literature. In the end, students appeared to draw on a range of knowledge about physical causality in the winch, for using algebraic equations (e.g., substituting values or solving), and for evaluating algebraic representations. This range of knowledge, and the ways students used it in the course of solving winch tasks, suggested a good match between my data and an epistemological perspective known as knowledge-in-pieces (diSessa, 1988, 1993). diSessa developed this perspective to explain emerging expertise in Newtonian mechanics. The perspective holds that knowledge elements are more diverse and smaller in grain size than those presented in textbooks. Growth and change consists of multiple, related processes including not only the construction of new knowledge elements but also the coordination of diverse knowledge elements and the extension or constriction of conditions under which particular elements may be applied productively. Thus, the final framing that has appeared in published reports of my dissertation (Izsák, 2000, 2003, 2004) evolved through continuous analysis of existing literature and over the course of two rounds of data collection and analysis.

**Discussion**

What from this example might be of use to doctoral students and junior faculty who are beginning to help students conduct their own research? First, in my view the term literature review is misleading and might be better described as an analysis of prior work. This means much more than summarizing the results of a set or sets of studies. It means identifying and summarizing main questions that bodies of research have addressed, theoretical perspectives and methods used in different bodies of research, and main results that have emerged across studies. It also means analyzing the possibly tacit assumptions underlying bodies of research and possible limitations that cut across related studies. Various handbooks (e.g., Berliner & Calfee, 1996; Grouws, 1992; Richardson, 2001) contain reviews by acknowledged experts in a wide variety of areas and are good places to start.

As my example illustrates, analyzing prior work can help identify problems for which you have a reasonable chance of making progress. Some problems, though important, may simply be too hard for a new researcher working individually. For instance, studies that have come the closest to constructing links between professional development for teachers and the learning of their students have been ambitious collaborative efforts among several seasoned researchers. High quality studies in a given area based on data that an individual researcher might reasonably collect and analyze can suggest an “approximate size” for a dissertation study. I stress the word approximate because published articles often report only a portion of a larger study, even a dissertation study.

Second, figuring out how to get data that contain warrants sufficient for making convincing claims is hard work. Pilot studies can help gauge the promise of a particular research design and can suggest refinements. I interviewed a wider range of students in the first year and narrowed in on Algebra I students the second as I searched for a good match between students and winch tasks. If I had not found a good match between students and winch tasks during the pilot, I would have needed different students, different tasks, or both. Pilot studies also provide an opportunity to work out kinks in data collection before the data really count. For instance, technically strong
videotaped data often requires experimentation with placements of cameras and configurations of microphones, especially in classrooms. Other aspects of research design can arise in studies using qualitative methods and the larger point applies to quantitative studies as well. For instance, a questionnaire or other instrument might need piloting and refinement.

Third, framing research is often a dynamic process that unfolds over the course of a study. This point is important because the process of framing research often differs from the logic authors present in published articles—the questions for the study, literature review, methods, analysis and results, and concluding discussion. By way of analogy, the final presentation of a proof in mathematics that works forward from givens does not necessarily reflect how the proof was constructed through initial efforts that led to dead ends, processes where the person worked forward from the givens and backwards from the conclusions hoping to connect the two strands of logic in the middle, and so on. The final elements in a coherent argument are often assembled through cycles of successive approximations that lead to increasingly refined and coordinated questions, theory, methods, and results. Thus, questions posed at the beginning of articles are not necessarily those with which studies were launched.

Finally, theoretical perspectives shape research questions, data collection, and analysis in ways that emphasize some aspects of phenomena under study while suppressing others. One way to understand more deeply how theories shape which aspects get emphasized and which get suppressed is to consider how the same phenomena might be investigated and explained using different theoretical perspectives. In my view, there are strong advantages to beginning studies with sets of alternative theoretical perspectives and to examining the extent to which each alternative does, or does not, help explain data. In my case, I started with two cognitive perspectives and then moved away from the process-object account when it clearly did not fit my data. Instead, the knowledge-in-pieces perspective was a good match because it allowed the forms and types of knowledge at play to be part of what was under investigation. To illustrate how a theoretical perspective can suppress some aspects of a phenomenon, had I begun my study committed to finding schemes and operations (e.g., Steffe, 2001), I might have overlooked comments that evidenced students’ criteria for algebraic representations. Furthermore, all of the perspectives I have discussed leave social aspects of learning in the background.

Because we are not always aware of ways in which theoretical perspectives shape what we attend to and “see” in data, considering different perspectives can help make the rationale for theoretical decisions more explicit and, in turn, facilitate stronger connections among all of the components essential to high quality research. Moments where all the components start to come together are exciting and make all the hard work very rewarding.

Author’s Note
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References


Pre-service Teachers’ Performance in their University Coursework and Mathematical Self-Efficacy Beliefs: What is the Role of Gender and Year in Program?

Mine Isiksal

This study investigates the effects of gender and year in program on the performance and mathematical self-efficacy beliefs of 145 pre-service mathematics teachers in Turkey. One of the main purposes of this study is to investigate how duration in a teacher education program influenced the performance and mathematical self-efficacy beliefs of pre-service teachers. In addition, gender differences between male and female pre-service mathematics teachers, depending upon year in program, are examined. Results revealed that there were significant statistical effects of gender and year in program on both pre-service teachers’ performance and self-efficacy scores. Female pre-service teachers scored significantly higher than males on performance, but no significant difference was detected between female and male pre-service teachers with respect to mathematics self-efficacy scores. Senior pre-service teachers had the highest scores compared to other students in the program on both performance and mathematics self-efficacy scores. Although the present study is small, the results tentatively suggest a further investigation of the relationship between performance and self-efficacy beliefs might be fruitful. Studying how mathematical self-efficacy develops across school years and what factors facilitate its development could yield valuable implications for the field of mathematics education.

Self-efficacy has been defined as an individual’s judgment of their capability to organize and execute the courses of action required to attain designated types of performances (Bandura, 1986, 1997). Social Cognitive theorists claimed that self-efficacy beliefs strongly influence the choices people make, the effort they expend, and the degree of anxiety they experience. Bandura (1977) pointed out that self-efficacy expectations are major determinants of whether a person will attempt a task, how much effort will be expended, and how much effort will be displayed in the face of obstacles. Similarly, Schunk (1989, 1991) reported that when students approach academic tasks, students with high self-efficacy work harder and for longer periods compared to students with a lower self-efficacy. The role of self-efficacy helps to explain why people’s performance attainment might differ even when they have similar knowledge and skills (Pajares & Miller, 1995). Bandura (1986) has suggested that personal self-efficacy is derived from four sources: (a) performance accomplishment, (b) vicarious experience, (c) verbal persuasion, and (d) emotional arousal. According to Bandura, actual experience, especially past success and failure, is the most influential source of efficacy.

Although researchers have examined the role of self-efficacy in various academic areas, mathematics has been a main focus (Hacket, 1985; Hacket & Betz, 1989; Pajares & Miller, 1994, 1995). Hackett and Betz (1989) defined mathematics self-efficacy as a situational or problem-specific assessment of an individual’s confidence in her or his ability to successfully perform or accomplish a particular mathematical task or problem. These researchers investigated the relationship among mathematical performance, mathematics self-efficacy, attitudes toward mathematics, and the choice of mathematics related majors by 153 college women and 109 college men enrolled in introductory psychology courses at a large Midwestern University. They reported that mathematics performance was correlated moderately with mathematics self-efficacy. Similarly, both mathematics performance and mathematics self-efficacy significantly and positively correlated with attitudes toward mathematics and mathematics related majors. Students with high scores on mathematics self-efficacy and mathematics performance tended to report lower levels of mathematics anxiety, higher levels of confidence, and a greater tendency to see mathematics as useful compared to those with low scores.

In another research study, Randhawa, Beamer and Lundberg (1993) who studied more than 225 high school students in Canada reported that mathematics self-efficacy was a mediator variable between mathematics attitudes and mathematics achievement.

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Preservice Teachers’ University Coursework Performance
Similarly, Stevens, Olivarez, Lan and Runnels (2004) evaluated self-efficacy and motivational orientation of students to predict mathematics achievement, including mathematics performance and students' plans to take additional mathematics courses. They studied 358 students from the 9th and 10th grades and reported that there was a relationship between self-efficacy and prior mathematics knowledge. In addition, self-efficacy predicts motivational orientation and mathematics performance. Similarly, Pietsch, Walker, and Chapman (2003) investigated the relationships among self-concept, self-efficacy, and performance in mathematics among 416 high school students, and self-efficacy beliefs were identified as most highly correlated with performance.

Although there are studies related to the mathematics self-efficacy beliefs of high school students, there are few studies that focus on pre-service teachers. It is believed that there is a need to assess the self-efficacious characteristics of teachers and their mathematics competencies before they go into the real classrooms. Teachers' sense of self-efficacy affects the effort they put into teaching, the goals they set, and their level of inspiration (Ashton, 1985). Thus, the factors that affect the mathematics self-efficacy beliefs of pre-service teachers will be one of the important concerns of this study.

A second important concern of the study will be gender differences in mathematics, which has been one of the most important issues studied in mathematics education for many years. A number of empirical studies have shown that males tend to outperform females in measurement, proportionality, geometry, spatial geometry, analytic geometry, trigonometry, and application of mathematics (Battista, 1990; Fennema & Carpenter, 1981) while females have performed better than males in computation, set operation, and symbolic relation (Fennema, 1974). Furthermore, female students outperform male students on assessment of mathematical ability at the elementary and middle school levels, whereas male students outperform female students at the high school and college levels (Aiken, 1986-1987; Hyde, Fennema & Lamon, 1990; Maccoby & Jacklin, 1974). On the other hand, in recent years and in most of the countries, including the United States, studies suggest that female and male students perform at similar levels in mathematics. The closure of this gap between male and female performance has been experienced in many countries. Ma (2004) investigated gender differences in all 43 countries involved in Programme for International Student Assessment 2000 by using data from the Organization for Economic Cooperation and Development (OECD). The results revealed consistent gender differences in favor of males in mathematics performance in most countries; however, he stated that these gender gaps in mathematics performance could be characterized as being universally small. In this sense, this study also aims to investigate whether these gender inequalities are still an issue in undergraduate programs in Turkey.

There is agreement that the relationship between gender and mathematics self-efficacy has not been explored as thoroughly as that between gender and math performance (Pajares & Miller, 1997). In their study with 327 eighth-grade students in a southern state of the USA, Pajares and Miller reported that females with low self-efficacy performed better than males with low self-efficacy, whereas males with higher self-efficacy performed better than females with high self-efficacy. Hackett and Betz (1981) mentioned that mathematics self-efficacy expectations of college males are stronger than those of college females, and, in a research study carried out with 262 undergraduate students enrolled in an introductory psychology course, males obtained significantly higher scores on mathematics self-efficacy scales. In comparison to females, males had a greater positive attitude toward mathematics, more confidence in their mathematics ability, and a greater tendency to view mathematics as more useful. Likewise, in a study with 350 undergraduates from a large public university, Pajares and Miller (1994) found that males reported higher mathematics self-efficacy than females, and females expressed higher levels of mathematics anxiety. Additionally, males had higher scores on the performance measure.

Contradictory to these findings, Cooper and Robinson (1991) reported no gender differences on mathematics self-efficacy, mathematics anxiety, and mathematics performance among undergraduates at a public mid-western university who selected mathematics oriented college majors. In another study, carried out by Schunk and Lilly (1984), the influence of performance on self-efficacy and attribution was investigated. Male and female students from grades 6 to 8 were asked to judge their self-efficacy for learning a novel mathematical task. Students were then provided with instruction and practice and received feedback. Although the females initially judged their self-efficacy as lower than the males, no gender difference was obtained at the end of the training.

As stated earlier, although there are studies that investigate the gender differences on mathematics
performance and mathematics self-efficacy at various grade levels, few studies focus on pre-service teachers. Having pre-service teachers with lower mathematics self-efficacy could affect their motivation, attitude, achievement, and even their future performance within the teaching profession. Thus, there is a need to assess pre-service male and female mathematics teachers' beliefs about their ability to perform specific mathematical competencies before they go into the classroom. In this respect, questions are raised whether gender and grade level are important factors that affect mathematics efficacy beliefs and performance of pre-service teachers during their university life. Thus, this study aims to examine how pre-service teachers' self-efficacy beliefs differ based on their year in the teacher education program and how gender differences affect these changes. Stated differently, this study aims to investigate the effect of gender and year in program on pre-service mathematics teachers’ performance and mathematical self-efficacy in Turkey.

Method

Participants

Data was collected from 145 freshmen, sophomore, junior, and senior pre-service middle school mathematics teachers enrolled in an undergraduate program at a public university in Ankara, Turkey. Participants’ age ranged from 18 to 24 and they were students within the Elementary Mathematics Teacher Education (ELE) Program in the department of education. Data were collected at the end of the spring semester of the 2003–2004 academic year. Participants of the study were asked whether they would voluntarily fill out a questionnaire. In total, 95 female and 50 male students participated in the study. The distribution of participants according to their year in the program is given in Table 1.

Table 1

<table>
<thead>
<tr>
<th>Grade Level</th>
<th>F</th>
<th>M</th>
<th>Total</th>
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<tr>
<td>Freshmen</td>
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<td>12</td>
<td>36</td>
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<tr>
<td>Sophomore</td>
<td>24</td>
<td>13</td>
<td>37</td>
</tr>
<tr>
<td>Junior</td>
<td>22</td>
<td>13</td>
<td>35</td>
</tr>
<tr>
<td>Senior</td>
<td>25</td>
<td>12</td>
<td>37</td>
</tr>
<tr>
<td>Total</td>
<td>95</td>
<td>50</td>
<td>145</td>
</tr>
</tbody>
</table>

Elementary Mathematics Teacher Education Program in Turkey

To be a university student in Turkey, senior high school students must pass the University Entrance Examination (UEE) administered by the Students Selection and Placement Center (ÖSYM) once a year. Based on their scores on the UEE, students are eligible to be students in the departments of universities in Turkey.

Participants of this study were pre-service elementary mathematics teachers in a four-year teacher education program in the department of education. Pre-service elementary mathematics teachers in a teacher education program plan to teach mathematics in middle schools (6th, 7th and 8th grades) and also in elementary schools (4th and 5th grades). In order to graduate from the Elementary Mathematics Education Program, pre-service teachers take mathematics and mathematics education courses, as well as Physics, Chemistry, English, Turkish, History, Statistics and courses from Educational Sciences. The mathematics and mathematics education courses taken by pre-service mathematics teachers include the courses offered by the Turkish Council of Higher Education.

Table 2

Mathematics and mathematics education courses taken by pre-service mathematics teachers

<table>
<thead>
<tr>
<th>Grade/Courses</th>
<th>Mathematics Lessons</th>
<th>Mathematics Education Lessons</th>
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<tbody>
<tr>
<td>Freshman</td>
<td>Analysis (I, II),</td>
<td>School Experience I</td>
</tr>
<tr>
<td></td>
<td>Abstract Mathematics, Geometry</td>
<td></td>
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<tr>
<td>Sophomore</td>
<td>Analysis (III, IV),</td>
<td>Curriculum Planning</td>
</tr>
<tr>
<td></td>
<td>Linear Algebra (I, II)</td>
<td>and Evaluation</td>
</tr>
<tr>
<td></td>
<td>Introduction to Algebra,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Analytic Geometry,</td>
<td>Instructional Technology and</td>
</tr>
<tr>
<td></td>
<td>Statistics and</td>
<td>Material Development,</td>
</tr>
<tr>
<td></td>
<td>Probability (I, II),</td>
<td>Special Teaching Methods I</td>
</tr>
<tr>
<td></td>
<td>Elementary Number</td>
<td>Computer Assisted Instruction</td>
</tr>
<tr>
<td></td>
<td>Theory</td>
<td>in Mathematics Education,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>School Experience II,</td>
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<tr>
<td></td>
<td></td>
<td>Special Teaching Methods II,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Evaluation of Subject Matter</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Course Books, Practice</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Teaching</td>
</tr>
<tr>
<td>Senior</td>
<td></td>
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Preservice Teachers’ University Coursework Performance
Education (YÖK) as given in Table 2 (YÖK, 1998a, 1998b).

**Instrument**

In order to determine the self-efficacy scores of pre-service teachers with respect to mathematics, the Mathematics Self-Efficacy Scale (MSES) developed by Umay (2001) was used. MSES is a Likert scale, using a five-point scale ranging from “Always =5” to “Never =1.” The score on MSES ranged from 14 to 70. The scale consists of 14 items, and these items were grouped under three dimensions based on a factor analysis. Umay (2001) stated that the items loaded in first factor are related to the mathematics self-perception. Items like “I can easily help people around, related to their mathematical problems” and “I realized that I’m losing my self-confidence while studying mathematics” were included in the first category. The second category, behavioral realization in mathematical topics, includes items such as “I feel competent enough in mathematical problem solving” and “I feel that I’m doing something wrong while solving mathematical problems.” The last factor related to transferring mathematics into daily life skills consists of items similar to “I think that I can use mathematics effectively in my daily life” and “I think mathematically while planning my time/day.”

In order to measure the internal consistency (reliability) of the MSES, Cronbach’s alpha was calculated. The Cronbach’s alpha for MSES was calculated as .83 with 145 pre-service teachers that is considered to be high in most social science applications.

MSES was applied to the freshman, sophomore, junior and senior pre-service elementary mathematics teachers. The instructions specified that students’ answers were anonymous, no answer was right or wrong, and students were supposed to choose the one answer per item that best described his or her opinion about the item. Completing the questionnaire required 10–15 minutes and the questionnaire was administered to students during their class hour. A total of 145 students answered the questionnaire.

Student performance was measured by their Cumulative Grade Point Average (CGPA) on a 4.00 scale. Students’ total CGPA at the end of the second semester of 2003–2004 academic year was used as a variable of their performance in this study. In other words, in this study mathematics performance refers to the cumulative grade point average of pre-service teachers on both mathematics and mathematics education courses.

**Results**

The descriptive statistics related to year in program of male and female students with respect to performances and mathematics self-efficacy scores are summarized in Table 3. Means, standard deviations, and sample size (N) are displayed for each dependent variable.

Multivariate Analysis of Variance (MANOVA) was conducted to determine the effect of gender and year in program on pre-service teachers’ performance and mathematics self-efficacy. In order to detect the

<table>
<thead>
<tr>
<th>Table 3.</th>
<th>Descriptive Statistics for male and female students with respect to self-efficacy scores and performance</th>
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<tbody>
<tr>
<td></td>
<td>Mean</td>
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<tr>
<td></td>
<td>Year</td>
</tr>
<tr>
<td>Performance</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
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<tr>
<td></td>
<td>4</td>
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<tr>
<td>MSES</td>
<td>1</td>
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<td></td>
<td>2</td>
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<td></td>
<td>3</td>
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</tr>
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significance in MANOVA, we were interested in Wilks’ Lambda and the associated probability value.

The results revealed that there was a statistically significant effect for both gender and year in program on pre-service teachers’ performance and self-efficacy scores (Wilks’ Lambda ($\lambda$) = 0.729, $F(6, 272) = 7.75$, $p < .05$). Multivariate test results showed that there was a significant mean difference on teachers’ performance and their self-efficacy scores with respect to both gender $F(2, 136) = 3.3$, $p < .05$ and year in program $F(6, 272) = 7.8$, $p < .05$. Univariate Analysis of Variances (ANOVA) on each dependent variable was conducted as a follow-up test to the MANOVA to reveal the effect of each independent variable on each dependent variable separately. The results showed that there was a significant effect of both gender ($p < .05$) and year in program ($p < .05$) on performance. In order to assess the importance of the findings, effect size, which indicates the relative magnitude of the differences between means, was calculated. Effect size is the “amount of the total variance in the dependent variable that is predictable from knowledge of the levels of the independent variable” (Tabachnick & Fidell, 1996, p.53). Eta Square ($\eta^2$), the commonly used effect size statistic, revealed that 25% of variance in performance was explained by year in program, and 5% by gender.

Since the overall F test was significant, follow-up tests were conducted to evaluate mean differences among grade levels in program. After carrying out the univariate analysis that showed there was a significant effect of both gender and year in program on performance, Bonferroni post hoc analysis was carried out to reveal the significant mean differences between grade levels on pre-service teachers’ performances. The results are given in Table 4.

Table 4
Mean difference of GPAs with respect to year in program

<table>
<thead>
<tr>
<th>DV</th>
<th>Grade I</th>
<th>Grade J</th>
<th>Mean Difference (I-J)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>14.0*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6.9*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>7.8*</td>
<td></td>
</tr>
<tr>
<td>GPA</td>
<td>3</td>
<td>1</td>
<td>6.1*</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.9</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>7.1*</td>
<td></td>
</tr>
</tbody>
</table>

* indicates the mean difference is significant at the .05 level

Bonferroni post hoc tests revealed there was a significant mean difference between senior and junior pre-service teachers ($p <.05$), between senior and sophomore pre-service teachers ($p <.05$), and between senior and freshman pre-service teachers ($p <.05$) respectively, where senior students had the highest performance scores and freshman had the lowest. Although sophomores scored higher, no significant mean difference between sophomore and junior students ($p > .05$) with respect to performance was found. On the other hand, there was a significant mean difference between juniors and freshmen ($p <.05$) and between sophomores and freshman pre-service teachers ($p <.05$) with respect to performance.

In Figure 1, the relationship between gender, year in program, and the gender and year in program interaction on performance is given. From the graph, it can be seen that female pre-service teachers had higher scores compared to males at each year in program. With respect to performance, senior pre-service teachers had the highest scores, and the mean difference between male and female seniors was smaller than in other grades. Gender differences were greater in freshmen and junior students compared to sophomore and senior students. Additionally from Figure 1, we can discuss the relationship between gender and year in program on performance.

From figure 1, we can conclude that there was no significant gender and year in program interaction on performance. In other words, the effect of gender did not depend on the year in program and females scored higher than males at each year in program.

Contradictory to these results, findings in terms of self-efficacy scores revealed that there was no significant effect of gender ($p >.05$) and year in program ($p >.05$) on mathematics self-efficacy scores. Eta Square ($\eta^2$)
revealed that gender explained 0.2%, and year in program explained 5% of the self-efficacy scores. In Figure 2, the relationship between gender, year in program, and self-efficacy is given.

**Figure 2.** The relationship between gender, year in program and mathematics self-efficacy

From Figure 2, similar to performance, senior pre-service teachers’ self-efficacy scores were the highest and junior pre-service teachers’ were the lowest compared to other grades. Freshman, junior, and senior females had higher scores compared to their male counterparts with respect to self-efficacy scores. However, sophomore males outperformed females with respect to self-efficacy scores. Results revealed that although there are gender differences between male and female pre-service teachers on their self-efficacy scores, these differences were not significant among grade levels. Furthermore, gender and year in program interaction was not significant with respect to self-efficacy scores, which implies the effect of gender did not depend on the grade level of students with respect to their mathematics self-efficacy scores.

To sum up the findings, we can say that senior pre-service teachers have the highest scores compared to other grades on both performance and mathematics self-efficacy. The difference between seniors and other grades is significant in terms of their performance. On the other hand, this difference is not significant with respect to self-efficacy scores. Results obtained in terms of gender differences were also similar, females outperformed males on both performance and self-efficacy scores except for the sophomores where males achieved higher scores than females on mathematics self-efficacy. Again, gender difference is significant in terms of performance, but is not significant in terms of mathematics self-efficacy scores.

**Discussion**

In this study, the overall test scores revealed that there was a significant effect of both gender and year in program on pre-service elementary mathematics teachers’ performance and mathematics self-efficacy scores. The fact that female pre-service teachers outperformed males on performance scores contradicts earlier studies that found male students outperformed female students at the high school and college levels (Aiken, 1986–1987; Hyde, Fennema & Lamon, 1990; Maccoby & Jacklin, 1974). Similarly, females’ higher self-efficacy scores were not consistent with the results that revealed a significant difference between male and female students on mathematics self-efficacy scores in favor of males (Betz & Hackett, 1983; Pajares & Miller, 1994).

In meta-analysis carried out by Hyde, Fennema, Ryan, et al., (1990) it was mentioned that gender differences in most aspects of mathematics attitude and affect were small. Although male students outperformed females on mathematics self-confidence, the difference was not significant. Similarly, Hyde, Fennema, and Lamon (1990) explained that gender differences in mathematics performance for all ages indicates only a slightly better performance by males. Ma (2004) found that these gender gaps in mathematics performance could be characterized as being universally small. In the present study, the higher performance and self-efficacy scores of females may be attributed to the strong correlation between the two constructs (Hacket & Betz, 1989; Randhawa et al., 1993; Stevets et al., 2004). Females’ higher competence in math related tasks could lead them to achieve higher scores on their performances or vice versa.

Similarly, the fact that female students significantly outperformed males on their performance measured by overall GPAs may be due to the cultural factors affecting attitudes of students toward mathematics and the teaching profession. In Turkey, similar to most countries, teaching is stereotypically seen as a female profession, especially at elementary and middle grade levels. The majority of mathematics and science teachers working in elementary and middle schools are female. Males, who more often have science or math majors, mostly prefer to work in secondary schools. After graduation from the program, pre-service teachers work in middle schools which...
could be an important factor on increasing motivation and efficacy-beliefs of female pre-service teachers’ to spend more time on their courses and achieve higher scores. In this study, results also revealed that there was no significant gender and year in program interaction on performance, which means that the effect of gender did not depend on the year in program. Regardless of the year in program, females scored higher than males with respect to performance. Similarly, in order to see whether the differences on self-efficacy scores among the year in program vary as a function of gender, we can discuss the gender and year in program interaction with respect to self-efficacy scores. Results also showed that this interaction was not significant. As in performance, the effect of gender did not depend on year in program. Thus, it can be concluded that cultural factors such as accepting teaching as a female profession and other constructs have positive effects for females. Female pre-service teachers outperformed their male partners throughout their university education with respect to performance and mathematics self-efficacy.

Geary (1996) differentiated between two sets of mathematical abilities, that he called them biologically primary and biologically secondary. Geary mentioned that both females and males have an innate set of biologically primary mathematical abilities such as numerosity, ordinality, counting, and simple arithmetic, which are numerical abilities, which is why no difference exists between young males and females. Contrary to the biologically primary abilities, secondary mathematical abilities arise only through interaction with the specific sociocultural practices. Such mathematical abilities include more complex and abstract domains of mathematics like algebra, geometry, and calculus. Geary stated that if there is no gender difference between males and females, there might be certain socio-cultural practices that influence the development of cognitive and affective systems supporting the biologically secondary mathematical abilities in males and females alike. In this study, since female students significantly outperformed male students, one possible explanation might be that socio-cultural practices discussed above might enhance the development of secondary mathematical abilities in females more than in males, supporting Geary’s view.

In addition to the gender differences, the results also revealed the significant effect of year in program on performance on pre-service mathematics teachers. Senior pre-service teachers scored significantly higher than junior, sophomore, and freshman students with respect to mathematics performance. Freshmen students’ lower grades may be attributed to their struggle to adapt to the program and university life after high school. Freshmen students in the program must take mathematics courses such as analysis, discrete mathematics, and other science classes with which they are not familiar. Similarly, senior students’ higher scores on performance may be attributed to their adaptation to the university life and to the program.

A number of mathematics classes offered by the program can be another source for senior pre-service teachers higher scores on performance compared to the other grades. As given in Table 2, senior students have no math or science classes in their final years if they have successfully completed their courses in previous years. Thus, students might feel more competent in their education courses compared to their mathematics courses. Attitude towards education courses, peer learning, career choice, or the idea that they will graduate soon, could also affect the performance of senior students by increasing their motivation and confidence, resulting in higher performance.

In terms of the self-efficacy scores, senior students again achieved the highest scores, where scores of students in other years in the program were similar to each other. Umay (2001) believed that the four sources from which self-efficacy derives (Bandura, 1977, 1997) are related to the content and application of mathematics and mathematics education courses. In other words, the mathematics self-efficacy beliefs of pre-service mathematics teachers’ stem from their competence in mathematics and also from their success in relation to the educational practices they are experiencing. Consequently, when we talk about the mathematics self-efficacy of pre-service teachers we should consider all the experiences throughout their program, including their work in mathematics and education courses. In this area, this study did make a connection between the performance and self-efficacy scores of pre-service elementary teachers. Senior students who had the highest performance also had the highest self-efficacy scores. Thus, a strong relationship between performance and self-efficacy scores can be deduced from these findings. Senior students’ greater mathematics self-efficacy beliefs or other affect variables like motivation, competence, and attitude might directly affect their performances or vice versa. Furthermore, Umay (2001) stated that education faculties in Turkey have been reconstructed and the teacher-training system has been standardized to follow one model throughout Turkey. In her study, she examined the perceived mathematics self-efficacy of the freshman and senior students in the division of
mathematics teaching. She found that there was a significant difference in terms of senior students with respect to mathematics self-efficacy. My results also support these findings where, although results are not significant, mathematics self-efficacy scores of senior students were higher than those of freshman students, which indicate that pre-service education programs can have an effect on the efficacy beliefs of students during their university education pre-service.

This study reveals that there is a statistically significant effect of gender and year in program on both pre-service teachers’ performance and self-efficacy scores. Senior pre-service teachers have the highest scores compared to the other lower graders and females outperformed the males at each year in program with respect to both performance and mathematics self-efficacy. In light of these findings, we can conclude that studies should be carried out with different grade levels to gain deeper insight into the cultural factors, attitudes, and affective variables that could create gender differences related to mathematics performance and mathematics self-efficacy. Further in-depth research studies should be carried out to explore how cultural factors, prerequisite knowledge and demographic characters can influence the development of self-efficacy beliefs of both pre-service and in-service teachers. In addition, how these variables relate and affect the instructional practices of teachers is warranted and essential. Investigating the reasons that can create differences among constructs including performance, self-efficacy beliefs, motivation, etc., studying how mathematics self-efficacy develops across the school years, and what factors facilitate its development could yield valuable implications for the educational field.

References


How Can Geometry Students Understand What It Means to Define in Mathematics?

Patricio Herbst
Gloriana Gonzalez
Michele Macke

This article discusses how a teacher can prepare the terrain for students to understand what it means to define a figure. Drawing on writings from mathematicians and mathematics educators on the role of definitions in mathematics, the authors argue that students develop a greater appreciation for the conciseness of a mathematical definition if they are involved in activities of generating figures that meet stipulated properties. The authors illustrate that argument with episodes from students’ play of a game called Guess My Quadrilateral! in two high school geometry classes.

Mathematical definitions can be described logically as the statement of the necessary and sufficient conditions that an object must meet to be labeled by a certain word or expression. Thus the expression “circle of center O and radius r” can be defined as the set of all points in a plane that are at a distance r from a given point O. This definition means: (1) that it suffices for a point to be at a distance r from O to be a point on the circle (or, no other condition is needed), and (2) that if a point is on the circle, its distance from O is necessarily equal to r (or, this condition cannot fail). In his essay on “Mathematical Definitions and Education,” however, Henri Poincaré (1914/2001) noted that compliance with logical stipulations is not enough:

A definition is stated as a convention, but the majority of minds will revolt if you try to impose it upon them as an arbitrary convention… Why should these elements be assembled in this manner? … What need does it fill? … If the statement is sufficiently exact to please the logician, the [answer to those other questions is what] will satisfy the intuitionist. But we can do better still. Whenever it is possible, the justification will precede the statement and prepare it. (p. 452)

Among the things that an intuitionist would appreciate about a definition, Poincaré noted, was the sense in which the object being defined was different that other neighboring objects:

The definition will not be understood until you have shown not only the object defined, but the neighboring objects from which it has to be distinguished, until you have made it possible to grasp the difference, and have added explicitly your reason for saying this or that in stating the definition. (Poincaré, 1914/2001, p. 452)

Those comments are especially appropriate with regard to students’ learning of definitions for geometric objects in the high school geometry class, suggesting that students will grasp what the definitions of particular geometric objects mean only if they also learn what it means to define a mathematical concept.

Definitions of Geometric Figures and Students’ Prior Experiences

Two elements of students’ prior knowledge seem to make this learning difficult. On the one hand, students come to us with some idea of what it means to define a word. These ideas are derived from their experiences in the highly verbal adult world. Students encounter many new words in their natural language as they read texts (take, for example, words like pollution or democracy). They wonder what those words mean and often relate to those words through the explanations of more competent speakers. In briefing them on what a word like pollution means, someone (or the dictionary, eventually) might try to spell out as much as can be said about the new word to foster understanding and proper usage, giving general statements, or alternative general statements, as well as particular examples of correct usage. If defining a word means spelling out what it means and enabling the audience to use it competently, it seems as though there is no reason to prefer succinct definitions. In that sense, mathematical definitions differ from the definitions of ordinary words.
On the other hand, students also come to us with a wealth of knowledge about geometric figures that they have been building since their toddler years and on through elementary and middle school. They have been naming figures by pointing to objects and using those words in geometric activity. They have a sense of familiarity with geometric figures that may conspire against our desire to develop in students the sense that definitions are needed. Indeed, if you started your first day of class asking your students to draw a circle, you would not very likely hear a student asking you “what do you mean by circle?” Words like circle, square, rectangle, or kite are familiar to them—to the point that students can use many of them without really questioning whether they know what those words mean. Therefore, as teachers of high school geometry, we could hardly claim that when we teach students the definitions of many of the geometric shapes, we are answering any question they might have as to what those words mean.

The instructional problem that we want to discuss derives from the foregoing discussion. Within the world of familiar objects and their names, the world of geometric figures, how can we create in students the sense that definitions are needed? Furthermore, how can students develop an appreciation for mathematical definitions—that is, statements that neither simply name nor describe beyond doubt but rather provide necessary and sufficient conditions for assigning a name to one kind of object and not to many others that could be similar to it in some respect?

Making Definitions: Descriptive and Generative Activities

The high school geometry course has usually been predicated as an opportunity to expose students to an example of a mathematical system, whereby they ought to learn to relate to figures not by what their names implicitly evoke but by what their definitions explicitly require. Edwin Moise expressed the following remark with which we agree:

The intuitive immediacy of geometric concepts has another important pedagogic consequence. One of the vital processes in creative mathematical work is the transition from an intuitive idea to an exact definition. (Moise, 1975, p. 476)

What kind of instructional activities can summon what children know about geometric figures and their names and at the same time give the teacher leverage for promoting an overhaul of what students understand by define? How can one satisfy Poincaré’s expectations, justifying the need for each of the stipulations involved in a definition and clarifying the differences between the object defined and its neighboring objects? Moise suggested that activities of describing the properties of a figure given through pictures could summon students’ intuitions of the figure being defined:

Nearly every geometric definition can be—and commonly is—preceded by a picture that conveys an intuitive idea. The definition can then be checked against the pictures, with a view to finding out whether the definition really describes the idea that it is supposed to describe. (p. 476)

One can probably play out a scenario in which students could take an active role in the description: The teacher shows several examples of rectangles—some large, some small, some with consecutive sides of very different length, some of very similar length. Then the teacher asks students to say what is common to all of the rectangles. If the students only mention properties that are common to parallelograms, the teacher may pull a parallelogram that is not a rectangle and ask students to comment on differences between the new shape and the ones on the board. This may strike some as a commonplace occurrence, for example, the social studies teacher who develops concepts of abstract ideas (e.g., democracy) by exposing his or her students to different examples and asking them to describe what they have in common might recognize that practice as familiar. And yet it may also strike the mathematics teacher as an unintelligent strategy in regard to how to make students aware of the difference between a description and a definition. Definitions are not descriptions—at stake in a definition is not just clarity on the use of a word but also succinctness of formulation. How can students develop an appreciation of the power of statements that balance clarity with succinctness?

We contend that to help students develop appreciation of the various aspects that Poincaré saw involved in the work of giving a definition, engaging them in the activity of describing figures is not enough. They also need to be engaged in activities of generating figures by the properties they should have. By generating (or prescribing) a figure we mean stipulating the conditions that a hypothetical figure would have to satisfy, and then finding whether a figure exists that satisfies such conditions. In this article we want to elaborate on that point, showing an example of an activity of generating a figure that we found useful in the process of engaging students in defining. We show an example of how we chose to
organize this kind of activity as we introduced students to the definitions of some of the special quadrilaterals.

The Quadrilaterals Unit and the “Guess My Quadrilateral!” Game

The teaching idea that we present was part of a unit that we designed and implemented in two accelerated geometry classes populated by a total of 53 students, mostly 9th graders, in a very large and diverse public high school. In planning a unit on quadrilaterals we posed to ourselves the instructional problem of how to create a context for students to think of the minimal conditions that a shape must satisfy in order to be one of the special quadrilaterals. We acknowledged the likelihood that students would understand what the special quadrilaterals were and set out to use (rather than ignore) that prior knowledge. We were actually able to confirm this prior knowledge using a questionnaire that we gave to the two classes before the unit (see Appendix A).

We observed that students knew quite a bit about squares, rectangles, and rhombi. For example, half of the students knew that diagonals of a rhombus are perpendicular and two thirds of the students knew that such was not the case for a rectangle. They also did not know some things—e.g., two thirds of the students did not think that diagonals of a rectangle are congruent—or had misconceptions—e.g., one third of the students asserted that diagonals of a rhombus are congruent. In planning the unit, we also expected that merely asking them to define those figures would not provide enough support for them to think of the differences between defining a mathematical object and defining an unknown word. The initial questionnaire actually confirmed the notion that students naturally associate the word definition with either ‘say as much as you know’ or ‘point to an object that bears this name.’ For a question that asked them to define a rectangle, we found that 88% of the students were evenly split among those that would provide too much information (e.g., a quadrilateral with two sets of parallel sides and four right angles) and those that would provide insufficient information (e.g., a quadrilateral with two sets of parallel sides), while only 12% provided definitions that were necessary and sufficient. It seemed as though students did have intuitions of what the figures are but if they were to appreciate mathematical definitions that provide necessary and sufficient conditions they would need help understanding in what sense one such statement is better than another one which says as much as one knows or one that names and points.

The lesson that we share here was part of a special unit on quadrilaterals that we designed and implemented over a period of three weeks. Rather than introducing one special quadrilateral at a time, only the parallelogram was studied on the first day of the unit. A few days after having created a long list of possible properties a quadrilateral could have, students were introduced to defining special quadrilaterals by way of a game that we called Guess My Quadrilateral! This game, we contend, provided an effective context for students to understand why a statement that provided necessary and sufficient conditions might be preferable to one that spells out as much as one knows. The game also created a context for students to deal with the whole neighborhood of special quadrilaterals and thus concentrate on what about each of them made it different from its neighbors. As a pedagogical strategy, the Guess My Quadrilateral! game is an example of an activity in which students are engaged in generating a figure. The play of the game itself and the discussions about playing that ensued eventually gave rise to the definition of each of the special quadrilaterals; this took most of two days. In the following paragraphs we describe what we planned and show examples of how students worked on it.

The game consisted of having each group of students determine the name of a shape they could not see. A generic representation of the figure had been drawn on a card that the teacher kept out of the students’ sight. Students could gather information about the unknown figure by asking the teacher questions that admitted only “yes” or “no” for answers. Groups played in parallel, each group asking the teacher questions about their shape; but groups competed against each other in being able to guess their own shapes while asking the minimal number of questions. Groups accrued 1 point per question they asked; and at anytime they decided they knew the shape, they could make a guess. Their questions could ask for any property from the list of possible properties they had generated some days before. In this way, students could ask questions like “does it have a right angle?” but not questions like “what are the measures of its angles?” (since the latter question could not be answered with a “yes” or “no”). Also, whereas it was okay for them to ask whether the unknown figure was a specific quadrilateral (e.g., “is it a rhombus?”), if the answer to such a question happened to be negative, the group would accrue 999 points—thus they were discouraged from blindly guessing and encouraged to make sure their eventual “guess” would just be a check against a near certainty. After each group had the
chance to play three times, the total scores for each group would be compared; and whichever group had made the least total number of points would win the game.

As far as managing the play of the game was concerned, to make sure students understood how to play, we had one group do a trial run of the game in front of the whole class. After that demonstration, we had students in their groups write their questions on easel pad sheets; and when they were ready to ask their first question, the teacher would visit the group, answer the question, and they would write that answer next to the question. At the end, after they had made a correct or incorrect guess about the identity of the figure, we would stick the picture of the previously unknown figure to the sheet. After all groups had guessed the identity of the figure, we would stick the sheets to the blackboard, enabling the students to see the ways in which people had played the game as a segue into definitions for the shapes.

The Game: An Example of Engaging Students in Generating a Figure

What is it about this game that may help create a context for discussion about definitions? The game is meant to engage students with figures in a different way than usual. The game makes students generate a figure rather than describe it, which we contend provides a meaningful context for students to understand what it means to define in mathematics. When students come to high school geometry, their familiarity with figures has been shaped by tasks that rely to a large extent on seeing the figures as they talk about them. In contrast, generating a figure relies on talking about the attributes of figures that cannot yet be seen; figures must be imagined on the basis of what the prescribed properties allow. The Guess My Quadrilateral! game helps convey the idea that the properties that are true of a figure are the ones that make a figure what it is. Furthermore the game provides a way to gauge the extent to which more information is needed in order to know what a figure is. By making students accrue points (which negatively affect their chance to win the game), the game allows students to realize that it might be a good idea to ask questions that add really important information. The existence of an actual card with the figure drawn on it helps ensure that whoever (the teacher or another student) responds to questions would do so fairly—in our case it helped students see the teacher as less of an oracle and more like a device of the game.

Learning to Define by Playing the Game

In playing Guess my Quadrilateral!, students confronted the need to stipulate conditions that would actually constrain what the unknown figure could be, differentiating it from all the other figures that it could possibly be without extra stipulations. As we saw them doing this, we recognized they were doing what Poincaré had identified as central in the work of defining—distinguishing an object from its neighboring objects. This thinking was visible as they discussed in their groups what kind of questions they should ask.

For example, a group composed of four students—Alana, Madeleine, Pavan, and Tobey—started debating whether they should ask a question that would determine if the unknown figure was a square; thus, they started going through the properties that would be true in the case of the square. First, Tobey suggested asking whether the figure had two sets of parallel sides, then Pavan suggested that they should ask whether all angles were congruent because if the response were negative they would be able to eliminate the square as well as the rectangle. Alana reaffirmed that suggestion by indicating that they "need to ask a question that will eliminate the most" shapes. They implemented this idea by asking whether all angles were congruent and used the response to decide where to go next. Thus, after a negative response they asked whether the figure had two pairs of opposite congruent angles; and after a positive response, they asked whether the sides were all congruent. Of the 13 groups that played the game in the two classes, five actually arrived at a rather complex decision tree for what question to ask at any given time, considering the responses to previous questions and converging to decisions on the unknown quadrilateral. (Figure 1, which was drawn to display the work turned in by the group of Heidi, Jessica, Mitchell, and Neil, gives an example of such a decision tree).

Students came to appreciate the cost and value of succinctness as they faced responses to complex questions. Groups did not always choose to ask questions that eliminated half of the available alternatives but rather questions that were engineered to identify one shape. Heidi, Jessica, Mitchell and Neil, for example, asked on one occasion whether the shape had “only one set of parallel sides” and praised themselves for their capacity to use the affirmative answer to guess that the shape was a trapezoid. On a different occasion, they also asked an overly restrictive question—whether the diagonals were perpendicular.
bisectors of each other—and the negative answer kept hidden from them a fact that they might have been able to get and use had they asked a simpler question. Since the unknown shape was a kite, it would have been helpful to ask, for example, whether one of the diagonals was a perpendicular bisector of the other one. So whereas they could see the benefit of getting at the defining property of a shape when they got the expected answer to a complex question, they could also understand how, on average, it would be advantageous to ask simpler questions, whose answers could provide useful information no matter what the answer was.

Some Advantages and Disadvantages of the Game

Clearly the game does not do all one would want in developing meaning for definitions. For example the game is somewhat neutral in regard to whether one should prefer defining special quadrilaterals as a hierarchy (whereby a parallelogram is a trapezoid; see Craine and Rubinstein, 1993) or as a set of disjoint categories (whereby parallellograms are not trapezoids). Mathematically there is little interest in defining these figures as disjoint categories; yet students could successfully play the game even if they did think that special quadrilaterals are disjoint categories.

Additionally, the game promotes the creation of general decision strategies such as lists of questions that will eliminate as many options as possible. Those do not always generate properties that would define a quadrilateral, even though they may be optimal for playing the game. Thus those groups that arrived at one such strategy, like the one shown in Figure 1, could generate a kite as a quadrilateral that does not have equal diagonals and does not have any sides that are parallel. The list of questions works to make a decision (given the available shapes), but the list of answers does not define a kite. Furthermore, a decision strategy based on the flowchart in Figure 1 would compel one toward defining a rhombus as a quadrilateral with all sides congruent and two pairs of parallel sides, but would not make it equally compelling to define it, say, as a quadrilateral with diagonals that are perpendicular bisectors of each other. Moreover, as Mitchell realized in his group, deciding on the order in which question are asked is very important as far as knowing what the answers mean. In that sense, the set of responses to the questions is different than the set of clauses in a mathematical definition—for the latter it is grammatical structure rather than an order that

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**Figure 1. A flowchart for deciding how to play the game**

2. Write a list of the best questions you should ask to guess the quadrilateral and pay few points.
indicates how the information each clause provides will be combined.

But the game was very successful in getting students to think about how much purchase they get with the stipulation of a condition. During the guessing of one figure in Heidi’s group, Mitchell commented that he had made a “stupid question” when he had asked if there was a pair of congruent angles. The teacher encouragingly said that the question was in fact a good one. But Mitchell responded, “How much does it rule out?” stressing the point that the quality of the question depends also on how much information it gives at the point of the game in which it is asked. A specific example of how students used this idea in playing is provided again by what happened in Alana’s group after they heard that the quadrilateral did not have all angles congruent and they went on with the question of whether the unknown figure had two pairs of congruent opposite sides. On account of having gotten an affirmative answer, Alana suggested that it could be a rectangle, but Madeleine quickly retorted that in that case all angles would be equal. Alana agreed, and she then suggested that the information they had at the time meant that the unknown figure could be “a rhombus or a parallelogram.” Similarly, when they decided that the next question would have to ask for the sides, Madeleine added “we know it’s not a square;” which they used later on to conclude, after finding out that sides were equal, that the figure was a rhombus. Thus students used the information gathered to decide not only what to ask next but also how to interpret the responses to the following questions. They could experience an important quality of a definition: that in saying what something is one is also saying what something is not.

**Using a Game like this to Teach Students the Definitions of Figures**

Playing the game Guess my Quadrilateral! does not substitute for spelling out definitions and, later on, formulating and proving properties of figures. The game is not even the only thing one should do before spelling out the definitions. The game is a great tool to activate previous knowledge, but it is not just that either. The game provides an important anchor for what it means to stipulate the various conditions that one puts in a definition, what each condition allows and what it rules out. In so doing, the game helps establish why one would want definitions to be succinct rather than verbose, even if one knew (as our students know) that many things are true about those concepts. Further, it helps students realize that alternative definitions could be provided for the same geometric figure: For example, students could be asked to compare the way the flowchart in Figure 1 generates the rhombus with a different set of questions.

In our case, we used the records of how people had played the game as a resource in getting students to formulate mathematical definitions for the shapes. Probably because the game had also activated students’ previous knowledge, when, after playing the game we asked students to make up definitions for the shapes, they did include some that were more like descriptions of all that they knew about a figure. Indeed, of the 13 groups, six responded to this task by providing verbose, kitchen-sink descriptions of quadrilaterals immediately after playing the game.

But the experience of having played the game made it possible and meaningful for the teacher to engage students in discussing a question that paved the way to making their definitions succinct: What do you have to know about a shape to make a successful call as to what that shape is? This question does not ask for a definition, but produces one—and can be held against other definitions that students might make—of the “description” or “name and point” nature. As the teacher told one of the classes, even though they might say “Oh, that’s important to know,” the game had helped them realize that such a property “maybe really [is] not that important and maybe I can say less and still figure out what shape it is.” After the class had looked at definitions for the special quadrilaterals only the work of one group exhibited a “definition” that could be more properly called a description. All of this stresses a point that Lakatos (1976) makes with his example of the concept of polyhedron: Definitions for mathematical ideas are shaped considering the theorems that could be proved thereafter.

The teacher also pointed out that definitions might differ. While for one group a particular property might be a part of the definition of the figure, for another group this property could be deduced from the definition of the figure. The game helped people understand why that makes sense—to the extent that a definition really is an efficient tool for intellectual (rather than visual) recognition, students could readily accept that a rhombus might be defined as a quadrilateral with four equal sides or as a quadrilateral whose diagonals are perpendicular bisectors of each other. They also understood that whatever one chose as the definition would condition not only what theorems one could formulate and prove but also what could be used in proving those theorems. Eventually, that seems to be the whole point in our insisting that students
know the definitions — not because there is a lot of doubt that they know what words mean but because the idea is to have students understand that knowledge as a system of connected propositions.

The *Principles and Standards for School Mathematics* (NCTM, 2000) proposes as a standard for instructional programs that students be given opportunities to “analyze properties and determine attributes of two- and three-dimensional objects; explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them” (p. 308). Along those lines, more important than knowing the facts of each figure is knowing how those facts are connected and how they can be organized as strong theorems derived from cleverly chosen definitions. Hence, it is not as much knowing the exact official definition that we should strive for as it is engaging students in the mathematical activity of defining—and for this, involving them in generating (or prescribing) figures might be a useful pedagogical tool.

**References**


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1 One could, however, later on, engage students in activities of generating figures that do not involve already drawn figures but rather require students to also find a model for their prescribed figure. We did not do that in the unit we taught at this time.

2 Names are pseudonyms.
Appendix A – Student Questionnaire

Quadrilaterals unit - Diagnostic assessment
Name:
Class period:
Date:

I. Check a box if the figure has the property

<table>
<thead>
<tr>
<th>Property</th>
<th>Square</th>
<th>Rectangle</th>
<th>Rhombus</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. It has two pairs of parallel sides</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Its diagonals bisect each other</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>3. Its diagonals are congruent</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>4. Its diagonals are also angle bisectors</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Its diagonals are perpendicular</td>
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</tbody>
</table>

II. Always-Sometimes-Never
Write A, S, or N in the Answer box, if you think the statement is Always true, Sometimes true, or Never true

<table>
<thead>
<tr>
<th>Answer</th>
<th>Statement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A rhombus has equal angles</td>
</tr>
<tr>
<td></td>
<td>A square is a rectangle</td>
</tr>
<tr>
<td></td>
<td>The diagonals of a rhombus make an obtuse angle</td>
</tr>
<tr>
<td></td>
<td>A rhombus is a kite</td>
</tr>
<tr>
<td></td>
<td>A rectangle has congruent diagonals</td>
</tr>
</tbody>
</table>

III. Definition and properties of rectangles

What is a rectangle?

What else do you think is true about the rectangle?

IV. Bisectors of a parallelogram
If you drew the four angle bisectors of a parallelogram, what could you say about the figure they make? Why?
Assessing the Impact of Standards-based Curricula: Investigating Students’ Epistemological Conceptions of Mathematics

Jon R. Star
Amanda Jansen Hoffmann

Since the advent of the NCTM Standards (1989), mathematics educators have been faced with the challenge of assessing the impact of Standards-based (or “reform”) curricula. Research on the impact of Standards-based curricula has predominantly focused on student achievement; here we consider an alternative: Students’ epistemological conceptions of mathematics. 297 participants were administered a Likert-scale survey instrument, the Conceptions of Mathematics Inventory. Of these, 163 had not experienced Standards-based curricula, while the rest had used a Standards-based curriculum for over three years. Our results indicate that students at the Standards-based site expressed more sophisticated epistemological conceptions of mathematics than those of the students from the non-Standards-based site. We interpret this result to suggest that implementation of Standards-based curricula may be having an effect on students’ epistemological conceptions of mathematics.

Since the advent of the National Council of Teachers of Mathematics’ (NCTM) Standards (1989, 2000), mathematics educators have been faced with the challenge of assessing the impact of Standards-based (or “reform”) curricula on the students who used them. (We use the term “Standards-based” and “reform” interchangeably to refer to curricula developed with support from the National Science Foundation to achieve the vision of the 1989 NCTM Standards.) Despite the complexity of this endeavor, there is a pressing need to evaluate what effect curricula emerging from Standards documents have had on the teaching and learning of mathematics. In prior research efforts evaluating reform curricula, impact has been defined and operationalized in several ways. Some evaluations of reform have defined impact broadly, to include a range of factors such as differences or changes in attitudes, beliefs, and achievement (e.g., Wood & Sellers, 1997). However, more recently, given the political climate of “what works,” research on the impact of curricula has exclusively focused on student achievement (e.g., Senk & Thompson, 2003).

While acknowledging that improvement in student achievement is one important way to evaluate the impact of reform curricula, in this study we examine the impact of reform curricula on students’ epistemological conceptions of mathematics. We present an analysis of students’ epistemological conceptions of mathematics in two different curricular settings to study the impact of Standards-based curricula.

Students’ Epistemological Conceptions of Mathematics

One’s beliefs and assumptions about the nature of knowledge and knowing establish a psychological context for learning. They color how the learner views a school subject and the process of coming to know in that subject matter. We refer to these ideas and assumptions as epistemological conceptions, which we define as students’ relatively unexamined beliefs and assumptions about the nature of knowledge and knowing that exist at varying levels of sophistication and commitment. Students’ epistemological conceptions of mathematics are suggested to be an important part of students’ experiences when learning and doing mathematics (Gfeller, 1999); they establish a psychological context for what it means to know and do mathematics (Schoenfeld, 1992). As epistemological conceptions are central to how students experience the learning process, they may be

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the single most important construct in educational research (Pajares, 1992).

Epistemological conceptions are a challenging construct to study, for at least three reasons. First, there are varying labels and conceptualizations used across the field of mathematics education and educational research more generally to refer to this construct. What we call epistemological conceptions, others refer to as epistemological beliefs (Hofer & Pintrich, 1997; Schommer & Walker, 1995) or epistemological stances (diSessa, Elby, & Hammer, 2002). We choose to use the term conceptions rather than beliefs because typically beliefs refers to students’ assumptions about knowledge and knowing in addition to a range of other constructs (e.g., self-efficacy beliefs); thus, we use conceptions to refer to the subset of beliefs that address epistemological issues. Similarly, we feel that epistemological beliefs is not an appropriate label, as this label has been used to refer to trait-like, relatively permanent ways of thinking. Further, another inappropriate term for our purposes is epistemological stances, which refers to either highly contextual ways of knowing (diSessa et al., 2002) or broader, paradigmatic orientations toward knowing (Schwandt, 2000). Our choice of the term epistemological conceptions indicates the domain of the construct, which focuses on participants’ ideas about the nature of knowledge and knowing, as well as an acknowledgement that these conceptions, as reported on a survey instrument, may exist at varying levels of commitment.

A second challenge inherent in studying students’ epistemological conceptions concerns whether conceptions are considered to be domain-specific (localized to the school subject of mathematics) or domain general (pertaining to a range of subject matters). In support of the latter view, Schommer and Walker (1995) found that students held a range of conceptions at consistently sophisticated levels across the domains of both mathematics and social studies, including conceptions of knowledge as less certain or simple, learning as a not quick process, and one’s ability as not being fixed. However, challenges to this work have been presented by Buehl, Alexander and Murphy (2002), who found evidence supporting the domain-specificity of certain conceptions (e.g., knowledge utility or value – mathematics is more related to other areas than history) when survey items were worded in reference to disciplines.

Finally, the potentially wide range of such conceptions further challenges any study of students’ epistemological conceptions. Conceptions can be interrelated in one’s mental structures in potentially limitless systems (Abelson, 1979). For example, beliefs about oneself as a learner could be related to beliefs about the learning process, which in turn could be related to one’s beliefs about mathematics as a domain. Also, beliefs about the self could also be supported by motivational beliefs. Given this wide range, it is unclear where to bound the study of epistemological conceptions or how to select focal conceptions for study.

Despite the challenges of conceptualizing this construct, a careful analysis of students’ epistemological conceptions of mathematics is critical, particularly in efforts to evaluate the impact of reform, for at least two reasons. First, epistemological conceptions have the potential to be the most salient or remembered aspect of students’ experiences in mathematics. It is our perspective that the conceptual and procedural knowledge of school mathematics tend to weaken over time, while broader conceptions of the nature of knowledge and knowing endure. This stance is supported by Bishop (1996), who states that affective factors “appear to survive longer in people’s memories than does conceptual and procedural knowledge, which unless it is regularly used tends to fade.” (p. 19). Similarly, McLeod (1992) suggests that conceptions orient students’ perceptions, which gives epistemological conceptions a significant role in shaping longer-term memories of mathematics. These remembered conceptions can be considered an indicator of the impact of reform as they are abstracted from students’ experiences in the mathematics classroom (Schoenfeld, 1992).

Second, epistemological conceptions are considered to orient learning of mathematics in terms of students’ motivation, achievement, and problem solving. For example, with respect to motivation, Stodolsky, Salk, and Glaessner (1991) suggested students’ conceptions about the nature of the school subject are related to their learning goals. Cobb (1985) demonstrated this relationship through two case studies of first grade students, illustrating that a student with an ego-involvement learning goal, such as a focus on performance, also held the conception that mathematical procedures were unrelated to one another. In contrast, a student with a task-involvement learning goal, or persistence in learning the material, viewed relations between procedures. Conceptions about the nature of the school subject co-occurred with particular learning goals for these students. Additionally, some conceptions are related to students’ academic achievement and problem solving behaviors.
Schoenfeld (1985, 1988) found that students who held a belief that mathematical problems should be able to be solved in 12 minutes or less (quick learning) also exhibited a lack of persistence when working on challenging problems. Quick learning has also been found to be the epistemological conception that was the strongest predictor of high school GPA—the less students believed in quick learning, the higher GPA they earned (Schommer, Calvert, Gariglietti, & Bajaj, 1997). More evidence is needed to explore whether and how additional epistemological conceptions may support or constrain students’ problem solving behaviors and achievement. However, as problem solving is often a significant activity in reform settings, exploring which epistemological conceptions are prevalent among students in reform settings could provide some initial support for future studies of relations between conceptions and students’ academic behaviors in mathematics.

**Students’ Conceptions in Standards-based Settings**

Relations between epistemological conceptions and students’ experiences with learning mathematics have been demonstrated empirically, but much of the existing research on epistemological conceptions in mathematics education focuses on teachers’ beliefs (e.g., Thompson, 1984; Cooney, Sheally, & Arvold, 1998) or was conducted before the widespread adoption of Standards-based curricula (e.g., Schoenfeld, 1988). Of the small number of studies focusing on students’ epistemological conceptions of mathematics in reform curricula, we briefly review three recent papers that have particular relevance to our work. These studies demonstrate the potential for comparing epistemological conceptions of students who experience different mathematics curricula.

Boaler (1998, 1999) compared students with similar demographic profiles and found that students’ experiences with different mathematics curricula impacted students’ epistemological conceptions. Boaler investigated high school students’ experiences in two curricular settings in England: one school with open-ended activities and the other with a more traditional textbook approach. She found that students in the open-ended setting were more likely to express enjoyment for doing mathematics and to appreciate thinking for themselves over memorizing. Students in the traditional setting were more passive about their learning, were more likely to have a set view of mathematics as a vast collection of exercises, rules, and equations, and viewed mathematics as relating less to the world than other school subjects.

Extending this line of work with high school students in the United States, Gresalfi, Boaler, and Cobb (2004) examined students’ experiences in three high school mathematics programs, one of which was more traditional than the other two, focusing on analyses of students’ epistemological conceptions. They determined that students who studied mathematics in a traditional setting expressed passive conceptions about learning mathematics, including seeing mathematics as an external domain of knowledge and not connecting with mathematics in a personal way. In contrast, students from reform classrooms expressed inquiring conceptions about learning mathematics, such that they were likely to use mathematics to ask questions and probe relationships they observed. The authors argued that these conceptions were more related to students’ experiences with curriculum than the learning preferences the students brought with them to the classroom.

Hofer (1999) also contrasted students (N = 438) in two different curricular settings, but she compared college undergraduates who experienced different forms of Calculus: one that emphasized active and collaborative learning both in and out of class and primarily focused on word problems, and the other a more traditional approach of lecture and demonstration. According to the author, students registered for calculus without knowing the type of instruction that would be utilized in their respective sections. The students in the non-traditional calculus course were found to have more sophisticated conceptions of mathematics; they were particularly less likely to believe that doing mathematics involves getting a right answer quickly. Achievement was positively correlated with sophistication in mathematical conceptions (as in Schommer et al., 1997), and students with sophisticated conceptions of mathematics were more likely to have mastery orientations to learning mathematics.

As these studies suggest, students who experience different mathematics curricula may develop different epistemological conceptions. The present exploratory study contributes to this growing line of research by examining the epistemological conceptions of students who have experienced different forms of mathematics curricula for extended periods of time. We introduce a forced choice assessment as an alternative to grounded qualitative analysis of conceptions (Boaler, 1998, 1999; Gresalfi et al., 2004). The choice to utilize a Likert-scale instrument was purposeful, rooted in our conceptualization of epistemological conceptions. Since we believe that students’ epistemological
conceptions exist at a relatively unexamined level, we asked students to offer their opinions with respect to particular statements. Forced choice instruments have been criticized for reflecting the researchers’ meaning making over the students’, but they also afford a relatively efficient method for collecting data on large numbers of students and allow for testing hypotheses (Schommer, 1998; Schommer et al., 1997; Schommer, Crouse, & Rhodes, 1992; Schommer & Walker, 1995).

**Method**

We explored the impact of *Standards*-based curricula on 297 students’ epistemological conceptions of mathematics. One hundred sixty three students who had not experienced *Standards*-based curricula, had been administered the survey as part of an initial study of the instrument in 1996. An additional 134 students, all of whom had experienced a *Standards*-based textbook series for over three years, were recruited as part of the present study to serve as a comparison group. (The details of these survey administrations are provided below.) Our hypothesis was that students who experienced different mathematics curricula would differ in their epistemological conceptions.

**Instrument**

Few survey measures have been developed for the purpose of studying secondary students’ epistemological conceptions of mathematics. One such measure is the Conceptions of Mathematics Inventory, or CMI (Grouws, 1994). The 56 questions on the CMI ask students whether they agree or disagree with certain statements about what it means to do, learn, and think about mathematics. The survey questions comprise seven scales (Composition of Mathematical Knowledge, Structure of Mathematical Knowledge, Status of Mathematical Knowledge, Doing Mathematics, Validating Ideas in Mathematics, Learning Mathematics, and Usefulness of Mathematics), each of which assesses a different aspect of students’ epistemological conceptions toward mathematics. Students respond on a 6-point Likert scale, with “1” expressing strong agreement and “6” expressing strong disagreement. A student who mostly agrees with all questions on the CMI would hold epistemological conceptions consistent with the aims of recent reform documents. Such a student would view mathematics as being composed of a useful, coherent, and dynamic system of concepts and ideas in which learning is accomplished by sense making and authority is found through logical thought. A student who mostly disagrees with statements on the CMI would find mathematics an irrelevant, unchanging collection of isolated facts and procedures, handed down from a book or teacher, which must be memorized.

Table 1 describes the seven scales of items on the CMI; see Appendix A for a complete list of the items on the CMI.

**Table 1**

<table>
<thead>
<tr>
<th>Scale of the CMI</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Composition of Mathematical Knowledge</td>
<td>Mathematical knowledge is composed of EITHER concepts, principles, and generalizations OR facts, formulas, and algorithms.</td>
</tr>
<tr>
<td>Structure of Mathematical Knowledge</td>
<td>Mathematics is structured EITHER as a coherent system OR a collection of isolated pieces.</td>
</tr>
<tr>
<td>Status of Mathematical Knowledge</td>
<td>Mathematics as EITHER a dynamic field OR a static entity.</td>
</tr>
<tr>
<td>Doing Mathematics</td>
<td>Doing mathematics is EITHER a process of sense-making OR a process of obtaining results.</td>
</tr>
<tr>
<td>Validating Ideas in Mathematics</td>
<td>Validating ideas in mathematics occurs EITHER through logical thought OR via mandate from an outside authority.</td>
</tr>
<tr>
<td>Learning Mathematics</td>
<td>Learning mathematics is EITHER a process of constructing and understanding OR a process of memorizing intact knowledge.</td>
</tr>
<tr>
<td>Usefulness of Mathematics</td>
<td>Mathematics is viewed as EITHER a useful endeavor OR as a school subject with little value in everyday life or future work.</td>
</tr>
</tbody>
</table>

Grouws, Howald, and Colangelo (1996) assessed 163 ninth, tenth, and eleventh graders on their conceptions with the CMI between 1995 and 1996. Their original study was not designed to take into account the impact of curricula on students’ epistemological conceptions, but the CMI authors recall that the study participants were exclusively selected from non-*Standards*-based mathematics classes in Missouri (Grouws, personal communication, July 13, 2001, October 5, 2001, May 20, 2003).
Although the questions on the CMI may appear to be designed to indicate whether students’ conceptions are consistent with the goals of reform, the CMI has never been administered to a large group of students in reform mathematics classes to “validate” its effectiveness. In other words, although many would assume that students with extensive experience in Standards-based mathematics would respond to the CMI in a manner different than those with extensive experience in non-Standards-based mathematics, we explored this assumption empirically. Grouws et al. provided us with mean score and standard deviations on each scale for each class in their original 1996 study; the original data from that study were not available.

Participants and Data Collection

As mentioned above, all 163 original respondents to the CMI (Grouws et al., 1996) came from non-Standards-based backgrounds in mathematics. Although no data were collected about the schools, courses, or instruction experienced by students in Grouws et al. original sample, our multiple and detailed conversations with the authors of the CMI make us confident that no student in this sample had any recent experience with NSF-funded reform-oriented curricula.

As a contrast, we sought to assess a sample of students who had a fairly long-term experience with Standards-based mathematics in a well-enacted setting. We recruited 134 9th grade students from a high school in Michigan to complete the CMI, early in their 9th grade year. (This survey administration took place between 2000 and 2001.) All students completed at least three years of reform-oriented instruction (in 6th, 7th, and 8th grades) in a middle school whose curriculum, the Connected Mathematics Project, (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1997), teachers, and pedagogy were quite familiar to us and, we believe, represented an extremely well enacted version of reform. (The lead mathematics instructor at this middle school is a professional development specialist for the Connected Mathematics Project; the Connected Mathematics Project has extensively documented her teaching as exemplary.) Students were administered the CMI in their regular mathematics classes by their teacher.

Note that while the two CMI administrations occurred approximately five years apart, we still conjectured that the curriculum would play a role in distinguishing differences between students’ conceptions, as recent research suggests the nature of curricula continues to play a key role in the development of students’ conceptions (Gresalfi et al., 2004), with students developing conceptions that are more aligned with inquiring stances toward mathematics in less traditional settings.

Reliability of Instrument

Grouws et al. (1996) do not report the statistical reliability of the CMI in their original study, but Walker (1999), who also used the CMI, reported that the CMI underwent a lengthy process of analysis and revision during its development in order to increase its reliability. Walker (1999) computed Cronbach’s alpha reliability values for her CMI data (N = 256 college students), which ranged from a low of 0.45 for the Composition and Doing scales to 0.91 for the Usefulness scale. We computed alpha reliability values using the CMI responses of the 134 students from a reform background; our results are given in Table 2. Our reliability values are comparable to Walker's (1999).

Table 2

Alpha Reliability of the 7 Scales of the CMI, N = 134

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<tr>
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<td>Learning</td>
<td>0.2603</td>
</tr>
<tr>
<td>Usefulness</td>
<td>0.8719</td>
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</table>

Nunnally (1978) and Litwin (1995) recommend Cronbach’s alphas close to or above 0.7 to indicate satisfactory internal consistency of constructs. By this standard, only the Usefulness scale of the CMI can be considered statistically reliable. However, it should be noted that neither Walker (1999) nor Grouws et al. (1996) qualified their results by expressing concern about the reliability of the CMI. We discuss the impact of the low reliability of the CMI scales below.

Results

Comparison of Students’ Epistemological Conceptions

We performed a t-test on the mean scores of each CMI scale, comparing Grouws et al. original sample of 163 students from a traditional background to our sample of 134 students from a reform background. Our results are shown in Table 3. Recall that all items used a six-point scale, and that the lower the number, the more reform-oriented the response.
We found that on average, students from a reform background responded reliably differently to the items on the CMI as compared to students from a traditional background. In particular, students’ responses in the reform setting were more aligned with reform-oriented ideas on the scales of the CMI than traditional students’ responses. The differences on each scale were statistically significant: Composition, $t(295) = 6.508, \ p < .001$; Structure, $t(295) = 10.3181, \ p < .001$; Status, $t(295) = 8.352, \ p < .001$; Doing, $t(295) = 11.754, \ p < .001$; Validating, $t(295) = 10.001, \ p < .001$; Learning, $t(295) = 11.043, \ p < .001$; and Usefulness, $t(295) = 13.933, \ p < .001$.

In addition to testing for significance, the magnitude of the differences between traditional and reform students’ responses can be seen by determining effect sizes using Cohen’s $d$ (Cohen, 1988). A value of Cohen’s $d$ larger than 0.8 indicates a large effect. Effect sizes are given in Table 4, indicating that there are very large differences between traditional and reform students’ responses on all scales.

The combination of a statistically reliable difference and a very large effect size on all scales leads us to conclude that, despite only moderate alpha reliabilities on most scales, students from a reform background responded differently on the CMI than did students from a traditional background. In other words, our data indicates that after experiencing several years of exemplary instruction in Standards-based mathematics curricula, students appear to develop epistemological conceptions of mathematics that are different from those from a more traditional background.

The results from the Usefulness scale are particularly compelling, as this scale was reliable, showed significant differences, and had a very large effect. The Usefulness scale items include, “Students need mathematics for their future work,” “Mathematics is a worthwhile subject for students,” “Students should expect to have little use for mathematics when they get out of school” (reversed item). The Cohen’s $d$ effect size of 1.33 for this scale indicates that the mean response of the reform students (3.50) was greater than the 90th percentile of the traditional group’s responses (Cohen, 1988). Reform students’ conception of the usefulness of mathematics was clearly quite different from that of students from a more traditional background.

Students’ responses on the Structure scale are also revealing. Although somewhat less reliable (see Table 2), this scale also showed significant differences and a very large effect. Structure scale items emphasize the inter-relatedness of ideas in mathematics; items included, “Mathematics involves more thinking about...
relationships ... than working with separate ideas,” “Most mathematical ideas are related to one another,” and “Mathematics consists of many unrelated topics” (reversed item). Reform students’ responses indicated a conception that relationships do exist between mathematical concepts. This is reminiscent of the work of Gresalfi et al. (2004), mentioned above, who found that students from classroom settings in which communication was emphasized were more likely to see the relationships between mathematical concepts and procedures.

Discussion

Our results contribute to the promising line of research demonstrating that Standards-based curricula may impact students’ epistemological conceptions of mathematics differently than traditional curricula (e.g., Boaler, 1998, 1999; Hofer, 1999; Wood & Sellers, 1997). Students who experienced different forms of mathematics curricula did indeed express different epistemological conceptions of mathematics, with high school students with experience in Standards-based curricula holding more sophisticated epistemological conceptions. These results are similar to Hofer’s (1999) study of college students. Of particular note is the difference on the Usefulness scale; students in Standards-based settings were much more likely to find mathematics to be useful, which is similar to Boaler’s (1998, 1999) findings.

Two limitations that temper these results are the lack of observations of the implementation of the curricula and our limited access to the Grouws et al. data set. First, in terms of implementation of the curricula, we acknowledge that we have only second-hand access to the traditional site. As mentioned previously, after communicating with Grouws and his colleagues, we are confident that the students used in their original study (Grouws et al., 1996) came from classrooms using non-Standards-based curricula; however, we have no data to confirm this. We recommend that future studies of students’ epistemological conceptions be complemented by classroom observations, as suggested by Hofer and Pintrich (1997).

Second, since we did not have full access to Grouws et al. (1996) data set, we were limited in the statistical analyses we could perform. As a result, we interpret the results of some of our scales with caution, given the moderate to low alpha scores for reliability. However, as discussed above, we did find significant differences and also large effect sizes on all scales. Future research could involve more sophisticated analyses (e.g., factor analyses) with the CMI to reassess and improve the scales for increased reliability.

It is our recommendation that efforts to assess the impact of Standards-based curricula broaden to include factors beyond student achievement. In order to achieve these ends, researchers should continue efforts to develop and refine instruments to assess large-scale groups of students. In order to further this line of research, researchers could work to extend the sample beyond two schools, making efforts to carefully match the sites, and analyze for alternative moderating variables, such as teaching style or student achievement. The conjecture that students’ experiences with Standards-based curricula could impact students’ dispositions toward mathematics, such as helping them develop more positive attitudes or sophisticated epistemological conceptions of mathematics, is worth further investigation. Teachers are invested in the development of students’ perspectives in addition to growth in their achievement; research should reflect this value.

Author’s Note

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References


Appendix A

Items on the CMI
(* Indicates reversed items)

Composition

9. While formulas are important in mathematics, the ideas they represent are more useful.
25. Computation and formulas are only a small part of mathematics.
39. In mathematics there are many problems that can’t be solved by following a given set of steps.
51. Essential mathematical knowledge is primarily composed of ideas and concepts.
*1. There is always a rule to follow when solving a mathematical problem.
*17. Mathematicians work with symbol rather than ideas.
*33. Learning computational skills, like addition and multiplication, is more important than learning to solve problems.
*49. The field of mathematics is for the most part made up of procedures and facts.

Structure

13. Often a single mathematical concept will explain the basis for a variety of formulas.
24. Mathematics involves more thinking about relationships among things such as numbers, points, and lines than working with separate ideas.
37. Concepts learned in one mathematics class can help you understand material in the next mathematics class.
50. Most mathematical ideas are related to one another.
*7. Diagrams and graphs have little to do with other things in mathematics like operations and equations.
*19. Finding solutions to one type of mathematics problem cannot help you solve other types of problems.
*31. There is little in common between the different mathematical topics you have studied, like measurement and fractions.
*41. Mathematics consists of many unrelated topics.

Status

11. New mathematics is always being invented.
27. The field of mathematics is always growing and changing.
42. Sometimes when you learn new mathematics, you have to change ideas you have previously learned.
54. Students can make new mathematical discoveries, as well as study mathematicians’ discoveries.
*3. When you learn something in mathematics, you know the mathematics learned will always stay the same.
*21. New discoveries are seldom made in mathematics.
*35. When you do an exploration in mathematics, you can only discover something already known.
*44. Mathematics today is the same as it was when your parents were growing up.

Doing

2. Knowing why an answer is correct in mathematics is as important as getting a correct answer.
16. When working mathematics problems, it is important that what you are doing makes sense to you.
32. Understanding the statements a person makes is an important part of mathematics.
56. Solving a problem in mathematics is more a matter of understanding than remembering.
*8. If you cannot solve a mathematics problem quickly, then spending more time on it won’t help.
*29. Being able to use formulas well is enough to understand the mathematical concept behind the formula.
*38. If you knew every possible formula, then you could easily solve any mathematical problem.
*48. One can be quite successful at doing mathematics without understanding it.
Validating

10. Justifying the statements a person makes is an important part of mathematics.
26. It is important to convince yourself of the truth of a mathematical statement rather than to rely on the word to others.
40. When two classmates don’t agree on an answer, they can usually think through the problem together until they have a reason for what is correct.
52. When one’s method of solving a mathematics problem is different from the instructor’s method, both methods can be correct.

*5. When two students don’t agree on an answer in mathematics, they need to ask the teacher or check the book to see who is correct.
*15. You know something is true in mathematics when it is in a book or an instructor tells you.
*28. You can only find out that an answer to a mathematics problem is wrong when it is different from the book’s answer or when the instructor tells you.
*45. In mathematics, the instructor has the answer and it is the student’s job to figure it out.

Learning

14. Memorizing formulas and steps is not that helpful for learning how to solve mathematics problems.
22. When learning mathematics, it is helpful to analyze your mistakes.
43. When you learn mathematics, it is essential to compare new ideas to mathematics you already know.
55. Learning mathematics involves more thinking than remembering information.

*4. Learning to do mathematics problems is mostly a matter of memorizing the steps to follow.
*18. Learning mathematics involves memorizing information presented to you.
*30. Asking questions in mathematics class means you didn’t listen to the instructor well enough.
*47. You can only learn mathematics when someone shows you how to work a problem.

Usefulness

6. Students need mathematics for their future work.
20. Mathematics is a worthwhile subject for students.
34. Knowing mathematics will help students earn a living.
46. Students will use mathematics in many ways as adults.

*12. Mathematics has very little to do with students’ lives.
*23. Taking mathematics is a waste of time for students.
*36. Mathematics will not be important to students in their life’s work.
*53. Students should expect to have little use for mathematics when they get out of school.
In Focus…

Deconstructing Teacher-Centeredness and Student-Centeredness Dichotomy: A Case Study of a Shanghai Mathematics Lesson

Rongjin Huang
Frederick K. S. Leung

Teacher-dominated classrooms with some student-centered elements are a perplexing phenomenon of Chinese mathematics classrooms. In-depth exploration of this phenomenon is helpful for understanding the features of mathematics teaching in China. This paper demonstrates how the teacher can encourage students to actively generate knowledge under the teacher’s control from a perspective of variation and further deconstruct the legitimacy of teacher-centeredness and student-centeredness dichotomy.

Teacher-dominated classrooms in countries under the influence of the Confucian-heritage culture (CHC) are often seen as an environment not conducive to learning in western countries (Biggs, 1996). However, students from CHC countries have consistently performed well in recent international studies of mathematics achievement (Beaton et al, 1996; Mullis et al, 1997; 2000; 2003). The mismatch between the unfavorable learning environment and the outstanding achievement has prompted discussion on the so-called “paradox of the Chinese learners” which led to many studies about the teaching in CHC classrooms and the psychological and pedagogical perspectives about Chinese teaching and learning (Leung, 2001; Watkins & Biggs, 2001; Fan, et al, 2004). To crack the paradox, some studies have tried to explore the mechanism of mathematics teaching in CHC settings (Huang, 2002; Huang & Leung, 2004; Mok, 2003; Mok & Ko, 2001). One interesting observation was made that there are some student-centered features in mathematics classrooms in CHC although the teaching is teacher-dominated. So, there are some elements of good teaching in the teacher-dominated classrooms in CHC (Watkins & Biggs, 2001), and a dichotomy of teacher-centeredness and students-centeredness may not be suitable to characterize whole classroom teaching (Huang, 2002; Mok & Ko, 2000). There is not a single teaching method guaranteeing students’ high achievement. Different countries share some common components of classroom teaching, but have different emphases and different combinations of those components (Hiebert et al, 2003). Although researchers have argued that there were many elements of student-centeredness in Chinese classrooms, there are seldom studies on how teachers encourage students to actively participate in mathematics learning in teacher-dominated classrooms (Huang, 2002; Mok & Ko, 2000; Mok & Morris, 2001). Looking at Chinese classrooms from a different perspective may shed new insights and understanding on what is really happening and whether it is conducive to student learning. A theoretical framework of variation was developed recently. It described how an enacted space of learning was constructed through creating certain dimensions of variation for students to experience (Marton & Booth, 1997; Mok, 2003; Marton et al, 2004; Gu et al, 2004). We will use this framework to analyze a Shanghai lesson and to demonstrate how students involve themselves in a process of learning, although the teacher controls the teaching.

**Theoretical Considerations**

According to Marton et al (2004), learning always involves an object of learning. The authors refer to the object of learning as a capability that has a general and a specific aspect. The general aspect has to do with the nature of the capability such as remembering, interpreting and grasping. The specific aspect has to do with the subject on which these acts of learning are carried out, such as formulas and simultaneous
equations. This object of learning is often conscious in the mind of the teacher and may be elaborated in different degrees of detail. What teachers are striving for is the “intended object of learning,” which is an object within the teacher’s awareness. However, what is more important is how the teacher structures the lessons so that it is possible for the object of learning to come to the fore of the students’ awareness, which is called the enacted object of learning (Mok, 2003). According to Marton et al (2004), learning is a process in which we want learners to develop a certain capability or a certain way of seeing or experiencing. In order to see something in a certain way the learner must discern certain features of the object. Experiencing variation is essential for discernment, and is thus significant for learning. Marton et al (2004) further argues that it is important to pay attention to what varies and what is invariant in a learning situation. Moreover, based on a series of longitudinal mathematics teaching experiments in China, and heavily influenced by cognitive science and constructivism, a theory of mathematics teaching/learning, called teaching with variation, has been developed (Gu et al, 2004; Huang, 2002). According to this theory, meaningful learning enables learners to establish a substantial and non-arbitrary connection between their new knowledge and their previous knowledge (Ausubel, 1964). Classroom activities are developed to help students establish this kind of connection by experiencing certain dimensions of variation. This theory suggests that two types of variation are helpful for meaningful learning (Gu et al, 2004). One is called “conceptual variation,” which provides students with multiple experiences from different perspectives. The other is called “procedural variation,” which is concerned with the process of forming a concept logically or chronologically, arriving at solutions to problems (scaffolding, transformation), and forming knowledge structures (relationship among different concepts). According to this theory, the space of variation, which consists of different dimensions of variation in the classroom, forms the necessary conditions for students’ learning in relation to certain objects of learning. For the teacher, the critical issue is how to create an adequate space of variation focusing on critical aspects of the learning object through appropriate activities. For the learner, it is important to be engaged in this ‘space’ of variation (called the enacted objective of learning). Gu et al. (2004) argued that by adopting teaching with variation, even with large classes, students could still actively involve themselves in the learning process and achieve meaningful learning. In this paper, this framework of variation is used to analyze a Shanghai lesson.

A Case Study

Setting and Data Source

Shanghai is one of the largest cities affiliated with the central government of the People’s Republic of China. A total area of approximately 6,000 square kilometers contains the population of 13 million people, which is about 1% of the population in China. It is a center of commerce and has a well developed education system. In Shanghai, the children start their schooling at the age of six. They receive 9 years of compulsory education. In Shanghai, there is a municipal curriculum standard which is different from the national one, a unified textbook according to the syllabus. Usually, candidates who finish Grade 12 could apply for a four-year full-time teacher’s training course, qualifying them to teach in secondary schools.

In this paper, a 40-minute videotaped lesson, which recorded the practice of a grade seven teacher working in a junior high school class in the countryside of Shanghai, constitutes the data source for this analysis. The videotape was an excellent lesson. The lesson illustrates well the teacher-dominated style of teaching, which is very common in China.

Description of the Lesson

The topic of the lesson is “corresponding angles, alternate angles, and interior angles on the same side of the transversal.” By and large, the lesson includes the following stages: review, exploration of the new concept, examples and practices, and summary and assignment.

Reviewing and inducing. At the very beginning, the teacher drew two straight lines crossing each other on the blackboard, and asked students to recall their learned knowledge such as concepts of opposite angles and complementary angles. When the teacher obtained the correct answers to those questions from the students, the teacher added one more straight line to the previous figure (see Figure 1(a)), and asked students how many angles there are in the figure, and how many of them are opposite angles and complementary angles. After that, the teacher guided students to explore the characteristics of a pair of angles formed by the two lines with the transversal by asking students the question: “What characteristics are there between \(\angle 1\) and \(\angle 5\)? [This actually is the new topic to be explored in this lesson].”
Exploring new concepts. In order to examine the relationship between $\angle 1$ and $\angle 5$, a particular diagram was drawn separately as shown in Figure 1(b). Through group discussion, the students found many features between these two angles, such as “$\angle 1$ and $\angle 5$ are both on the right hand side of line 1, and both are above the relevant line (line a and line b).” Based on students’ responses, the teacher summarized the students’ explanations and stated the definition of the “corresponding angles.” Then, students were asked to identify all the “corresponding angles” in Figure 1(b). Similarly, the two concepts “alternate angles” and “interior angles” were explored respectively.

Example and exercise. After introducing the three angle relationships, students were asked to identify them in different configurations. The problems are as follows:

Task 1: Find the “corresponding angles, alternate angles, and interior angles on the same side of the transversal” in Figure 2.

Task 2: Find the “corresponding angles, alternate angles, and interior angles on the same side of transversal” in Figure 3.

Task 3: In Figure 4, please answer the following questions: (1) Is $\angle 1$ and $\angle 2$ a pair of corresponding angles? (2) Is $\angle 3$ and $\angle 4$ a pair of corresponding angles?

Task 4: Given $\angle 1$ is formed by lines 1 and a as shown in Figure 5. (1) Please draw another line b so that $\angle 2$ formed by lines 1 and b and $\angle 1$ is a pair of corresponding angles. (2) Is it possible to construct a line b so that $\angle 2$ formed by lines 1 and b is equal to $\angle 1$?

Summary and assignment. The teacher emphasized that these three types of relationship are related to two angles located at different crossing points. These angles are located in a “basic diagram” which consists of two straight lines intersected by a third line. The key to judging these relationships within a complicated figure is to separate out a proper “basic diagram” which includes the angles in question. Moreover, the teacher demonstrated how to remember these relationships by making use of different gestures.

Finally, some exercises from the textbook were assigned to students.

Enacted Object of Learning

From the perspective of variation, and in order to examine what learning is made possible, we need to identify the dimensions of variation. In the following section, we look at the lesson in greater detail from this particular theoretical perspective focusing on the enacted object of learning and the possible space of learning.

Reviewing and Inducing

At the first stage of the lesson, a variation was created by the teacher through demonstration and questioning: varying two straight lines crossing each other to two straight lines intersected by a third one. By opening with this variation, the relevant previous knowledge was reviewed and the new topic was introduced in a sequential and cognitively connected manner. Thus, this variation is a procedural variation.

The next stage of the lesson was a stage of introducing and practicing new concepts. Two dimensions of variation were alternately created which are crucial for students to generate an understanding of the new concepts.

Representations of New Concepts

During the process of forming new concepts, the representations of the new concepts have been shifted among the following forms: rough description, intuitive description, definition, and schema. After a group discussion, the students were invited to present their observations, and based on students’ explanations, the new concepts were built through teacher guidance, finally, the concepts were imbedded in a “basic diagram,” i.e. “two straight lines intersected by a third line” (see Figure 1).

Different Orientations of the “Basic Diagram”

After the concepts of three types of angle relationship were constituted in a “basic diagram,” the teacher provided students with Task 1. By doing so, a new dimension of variation was opened for students to experience how to identify those relationships in different orientations of the figures. The teacher purposely varied the figures in position and number of angles in the figures (see Figure 2).
By providing students with these variations, the students were exposed to the concepts from different orientations of the diagram, which may make students aware that these concepts are invariant even when the orientation of a diagram is varied.

Figure 2. Variations of the basic diagram.

Different Contexts of the “Basic Diagram”

After students received a rich experience of these concepts in terms of the different orientations of the basic diagrams, the teacher then provided a group of tasks in which the “basic diagrams” were embedded in complex contexts (Task 2 and Task 3). Through identifying the angle relationships in different contexts of the “basic diagram,” an invariant strategy of problem solving, i.e. identifying and separating a proper “basic diagram” from complex configurations, came to the fore of students’ awareness. In general, separating a proper sub-figure from a complex figure is a useful strategy when solving a geometric problem (see Figure 3).

Different Directions for Applying the New Concepts

As soon as the students answered Task 2 (see Figure 3), the teacher posed a new and challenging question: “Conversely, if $\angle 1$ and $\angle 5$ are a pair of corresponding angles, which basic diagram contains them?” After students were given some time to think about the question, one of them was nominated to answer the question. The student gave a correct answer by saying that the basic diagram is “straight line a and b intersected by straight line c.” Similarly, by searching for a pair of interior angles on the same side of the transversal of $\angle 3$ and $\angle 12$, students identified a basic diagram, “straight lines c, d intersected by straight line a.” Once the students completed the above questions, the teacher summarized that the key points for solving these problems is to pick out a “basic diagram” (for instance, two straight lines a, and b intersected by a third straight line c) by deliberately covering up one line (d) from the figure. Through identifying the three angle relationships within basic diagram or separating relevant ‘basic diagrams’ so that the given angle relationship is true, the students not only consolidated the relevant concepts, but more importantly, learned the separation method of problem solving as well, i.e. separating a basic sub-figure from a complex configuration.

Contrast and Counter-example

After the preceding extensive exercise, the students might believe that they had fully mastered the focus concepts. At this point, the teacher posed Task 3 (see Figure 4) to test whether they had really mastered the concepts and methods of problem solving. Through separating a basic diagram as shown in Figure 6(a), students were asked to justify that “$\angle 1$ and $\angle 2$ are

Figure 3. “Basic diagram” embedded in complex contexts.

Figure 4. What are the corresponding angles?

Figure 5. Can you draw another line b so that $\angle 2$, formed by lines 1 and b, is equal to $\angle 1$?
corresponding angles.” However, since students could only identify a diagram as shown in Figure 6(b), they failed to see that “$\angle 3$ and $\angle 4$ are a pair of corresponding angles.” Thus a new dimension of variation of experiencing corresponding angles was opened: example or counter-example of the visual judgment.

**Figure 6. Example and counter-example.**

**Creating a Potential for Learning a New Topic**

After solving the above problems through observation and demonstration, the teacher presented a manipulative task (Task 4). First, through playing with colored sticks, the first question of task 4 was solved (see Figure 7(a)). Then, based on drawing and reasoning, the second question was also solved (see Figure 7(b)). During the process of problem solving, *the students’ thinking was shifted* along the following forms: concrete operation (by playing with the colored sticks) (enactive); drawing (iconic); and then logical reasoning (abstract). This exercise had two functions. On the one hand, the “previous proposition: opposite angles are equal” was reviewed; on the other hand, “a further proposition: if the corresponding angles are equal, then the two lines are parallel” was operationally experienced. That means a potential space of learning was opened implicitly.

**Consolidation and Memorization of the Concepts**

As soon as the key points for identifying three angle relationships in a variety of different situations were summarized, the teacher skillfully opened a new variation by making use of some gestures of the fingers (see Figure 8) in helping students to memorize the two concepts: mathematical concept and physical manipulation. This experience is helpful for visualizing and memorizing mathematical concept.

**Figure 7. Construction corresponding angles.**

**Figure 8. Demonstration angle relationships by gestures.**

**Summary and Discussion**

**Summary**

According to the theoretical perspective, it is crucial to create certain dimensions of variation bringing the enacted object of learning to students’ awareness. These objects of learning can be classified into two types. One is the content in question (such as concepts, propositions, formula), and the other is the process (such as formation of concepts, or process or strategy of problem solving). Based on the categories of variation: conceptual variation and procedural variation, it was demonstrated that conceptual variations served the purpose of building and understanding concepts, while procedural variations were used for reviewing previous knowledge and introducing the new topic, consolidating new knowledge, developing a strategy to solve problems with the new knowledge, and preparing for further learning implicitly. These two dimensions of variation were created alternatively for different purposes of experiencing the enacted objects of learning (see Table 1).
The aforementioned lesson unfolded smoothly, strictly following a deliberate design by the teacher. It is likely that it would be labeled as a teacher-dominated lesson from a Western perspective. However, if students’ involvement and contribution to the creation of these variations (i.e. enacted object of learning) are taken into consideration, it is hard to say that students are passive learners. This paper intends to demonstrate that the teacher can still encourage students to actively generate knowledge through creating proper and integrated dimensions of variation although the whole class teaching is under the teacher’s control. Thus, it seems to suggest that creating certain dimensions of variation is crucial for effective knowledge generation in large classrooms. Wong, Marton, Wong, & Lam (2002) argued that in teaching with variation, the space of dimension of variation constituted jointly by the teacher and the students is of crucial importance for understanding what the students learn and what they cannot possibly learn.

In order to re-conceive the dichotomy of teacher-centeredness and student-centeredness, Clarke and Seah (2005) adopted a more integrated and comprehensive approach, by analyzing both public interactions in the form of whole class discussion and interpersonal interactions that took place between teacher and student and between student and student during between-desk-instruction. They found that the style of teaching in both Shanghai schools was such that the teachers generally provided the scaffold needed for students to reach the solution to the mathematical problems without “telling” them everything. Hence, one could find quite a few math-related terms, which the teacher had not taught, that were introduced by the students during public discussion. The practices of the classroom in Shanghai sample school provided some powerful supporting evidence for the contention by Huang (2002) and Mok and Ko (2000) that the characterization of Confucian-heritage mathematics classrooms as teacher-centered conceals important pedagogical characteristics related to the agency accorded to students; albeit an agency orchestrated and mediated by the teacher.

A unique teaching strategy consisting of both teacher’s control and students’ engagement in the learning process emerges in Chinese classrooms. (Huang, 2002, p. 227)

Once the distribution of responsibility for knowledge generation is adopted as the integrative analytical framework, the oppositional dichotomization of teacher-centered and student-centered classrooms can be reconceived as reflecting complementary responsibilities present to varying degrees in all classrooms (Clarke, 2005).

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<td>April 24-26</td>
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<td>National Council of Supervisors in Mathematics</td>
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<td>NCTM</td>
<td>St. Louis, MO</td>
<td>April 26-29</td>
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<td>National Council of Teachers of Mathematics</td>
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<tr>
<td>History of Mathematics and Teaching of Mathematics</td>
<td>University of Miskolc,Hungary</td>
<td>May 18-21</td>
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<tr>
<td>CAMESG</td>
<td>Calgary, Canada</td>
<td>June 3-7</td>
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<td>Canadian Mathematics Education Study Group</td>
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<tr>
<td>ICTM3</td>
<td>Istanbul, Turkey</td>
<td>June 30 – July 5</td>
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<td>Third International Conference on the Teaching of Mathematics</td>
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<td>PME-30</td>
<td>Prague, Czech Republic</td>
<td>July 16–21</td>
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<td>International Group for the Psychology of Mathematics Education</td>
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<td>JSM of the ASA</td>
<td>Seattle, WA</td>
<td>August 6–10</td>
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<td>Joint Statistical Meetings of the American Statistical Association</td>
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<td>GCTM</td>
<td>Rock Eagle, GA</td>
<td>October 19–21</td>
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<td>Georgia Council of Teachers of Mathematics Annual Conference</td>
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